INSTITVTIONVM
CALCVLI INTEGRALIS
VOLVMEN SECVNDVM

IN QVO METHODVS INVENIENDI FVNCTIONES VNIVS VARIABILIS EX DATA RELATIONE DIFFERENTIALIVM SECVNDI ALTIORISVE GRADVS PERTRACTATVR.

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CALCVLI

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LIBER PRIOR.

PARS SECVNDA
SEV
METHODVS INVENIENDI FVNCTIONES VNIVS VARIABILIS EX DATA RELATIONE DIFFERENTIALIVM SECVNDI ALTIORISVE GRADVS.

SECTIO PRIOR
DE
RESOLVTIONE AEQVATIONVM DIFFERENTIALIVM SECVNDI GRADVS DVAS TANTVM VARIABILES INVOLVENTIVM.

CAPVT I.

DE
INTEGRATIONE FORMVLARVM
DIFFERENTIALIVM SECVNDI GRADVS SIMPLICIVM.

Definitio.

706.

Positis binis variabilibus x et y, si vocetur $dy = pdx$ et $dp = qdx$, aequatio quaecunque, relationem inter quantitates x, y, p et q definiens, vocatur aequatio differentialis secundi gradus inter binas variabiles x et y.

Coroll. 1.

707. Quemadmodum ergo littera p implicat rationem differentialium primi gradus, dum est $p = \frac{dy}{dx}$, ita littera $q = \frac{dp}{dx}$ implicat rationem differentia-

rentia-

rentialium secundi gradus. Sumto enim vt vulgo fieri solet elemento dx constante, erit $dp = \frac{ddy}{dx}$ ideoque $q = \frac{ddy}{dx^2}$.

Coroll. 2.

708. Quatenus ergo in aequatione proposita littera q inest, eatenus ea est differentialis secundi gradus. Si enim q abesset, ob solam p esset tantum differentialis primi gradus; ac si neque p neque q inesset, aequatio foret inter x et y, neque quicquam praeterea quaereretur.

Coroll. 3.

709. Methodus ergo desideratur, proposita aequatione quacunque praeter binas variabiles x et y etiam quantitates ~~$p = \frac{dy}{dx}$ et $q = \frac{dp}{dx}$ inuoluente~~, inveniendi relationem inter ipsas x et y, vnde pateat, qualis y sit functio ipsius x seu vicissim.

Scholion 1.

710. Hoc modo litteram q introducendo aequationes differentio-differentiales a conditione illa, qua quodpiam differentiale primi gradus pro constante assumi solet, liberantur. Cum enim ad meras quantitates finitas reuocentur, quae rationem differentialium primi gradus exprimunt, consideratio differentialis constantis ne locum quidem habere potest.

poteſt. Quando ergo aequationes differentio - differentiales more ſolito ita exhibentur, vt quodpiam differentiale conſtans ſit aſſumtum, introducendo litteras $p = \frac{dy}{dx}$ et $q = \frac{dp}{dx}$, ſpecies differentialium penitus tollitur, dum aequatio tantum quantitates finitas complectitur. Atque etiam viciſſim propoſita aequatione inter quantitates finitas x, y, p, q ea ad formam vulgarem infinitis modis reduci poteſt, prout aliud atque aliud differentiale pro conſtante aſſumitur, quae tamen omnes formae ſpecie diuerſae inter ſe perfecte conueniunt, quin etiam nullo differentiali conſtante aſſumto euolutio in formam ſolitam fieri poteſt.

Scholion 2.

711. Primum igitur breuiter exponi conueniet, quomodo aequatio more ſolito per differentialia ſecundi gradus expreſſa ad formam noſtram reduci queat, quodcunque differentiale conſtans fuerit aſſumtum. Sit ds hoc differentiale pro conſtante ſumtum, cuius ergo ratio ad dx, ob $\frac{dy}{dx} = p$, per p et forte ipſas variabiles x et y datur; ponatur ergo $ds = v dx$, vt v fiat quantitas finita. Iam cum in aequatione occurrant ddx et ddy, vel alterutrum ſaltem loco ddx ſcribatur $ds d.\frac{dx}{ds}$, quia ob ds conſtans fit vtique $ds.d.\frac{dx}{ds} = ddx$. Erit ergo $ddx = ds\,d.\frac{1}{v} = -\frac{ds\,dv}{vv}$. Simili modo loco ddy ſcribendo $ds.d.\frac{dy}{ds} = ds\,d.\frac{p}{v}$;

 fiet

fiet $ddy = \frac{dx(vdp - pdv)}{vv}$. Cum igitur v per p, x et y detur, erit

$$dv = Mdx + Ndy + Pdp = dx(M + Np + Pq)$$

ob $dp = qdx$, ficque fiet:

$$ddx = -\frac{dx^2}{v}(M + Np + Pq) \text{ et}$$

$$ddy = \frac{dx^2}{v}(qv - Mp - Np^2 - Ppq)$$

hique valores loco ddx et ddy fubftituti in aequatione tantum differentialia primi gradus relinquent, quae omnia ad dx reducta, aequatio per diuifionem prorfus a differentialibus liberabitur. Deinde viciffim huiusmodi aequatio inter x, y, p et q propofita in formam folitam fumto quopiam elemento ds conftante euoluetur, fi primo pro p vbique fcribatur $\frac{dy}{dx}$, loco q autem $\frac{1}{dx}d.\frac{dy}{dx} = \frac{dxddy - dyddx}{dx^3}$, vbi quidem nullius adhuc elementi conftantis ratio eft habita. At ob $ds = vdx$ conftans, infuper erit

$$vddx + dvdx = 0, \text{ feu ob } dv = Mdx + Ndy + Pd\frac{dy}{dx}$$

$$vddx + Mdx^2 + Ndxdy + \frac{P(dxddy + dyddx)}{dx} = 0$$

vnde pro lubitu vel ddx vel ddy elidi poteft, neutro autem elifo infinitae formae aequiualentes exhiberi poffunt.

Scholion 3.

712. Hinc ergo praeftantia formae finitae, ad quam hic aequationes differentio - differentiales reuocamus

eamus prae more ſolito eas exhibendi luculenter perſpicitur; cum eadem aequatio more ſolito infinitis modis, prout aliud atque aliud elementum conſtans aſſumitur, repraeſentari poſſit, dum noſtro more eadem aequatio ſemper ad vnicam formam reducitur. Quodſi ergo noſtro more aequationes prodeant diuerſae, certum eſt iis quoque diuerſas relationes, inter variabiles x et y exprimi, cum contra ſolito more diuerſiſſimae aequationes differentio-differentiales eandem relationem indicare queant, ex quibus plerumque difficile eſt eam eligere, quae ad reſolutionem maxime ſit accommodata. Cum igitur hic eiusmodi methodus requiratur, cuius ope propoſita quacunque aequatione inter quaternas quantitates x, y, p et q relatio inter binas variabiles x et y definiri queat, quoniam haec quaeſtio vires humanas ſuperare videtur, a caſibus ſimpliciſſimis erit exordiendum. Caſus autem ſimpliciſſimi ſine dubio ſunt, quando in aequatione propoſita duae tantum inſunt quantitates, ſcilicet vel x et q tantum, vel y et q vel p et q; hoc eſt ſi q aequetur functioni vel ipſius x, vel ipſius y vel ipſius p tantum; quos caſus in hoc capite euoluere conſtituimus.

Definitio.

713. Formula differentio-differentialis ſimplex eſt, quando poſito $dy=pdx$ et $dp=qdx$, quantitas q aequatur functioni vel ipſius x vel ipſius y, vel ipſius p tantum.

Coroll. 1.

Coroll. 1.

714. Triplices ergo habemus formulas differentio-differentiales ſimplices, quarum reſolutionem in hoc capite doceri conuenit, prout quantitas q vel per functionem ipſius p, vel ipſius x, vel ipſius y tantum determinatur.

Coroll. 2.

715. Si ergo X denotet functionem ipſius x, Y ipſius y et P ipſius p tantum, terna genera harum formularum ſimplicium ſunt 1) $q = X$; 2) $q = Y$; 3) $q = P$ in quibus continetur caſus ſimpliciſſimus $q = Const.$

Coroll. 3.

716. Si has formulas more ſolito exprimere velimus ob $q = \frac{dp}{dx} = \frac{1}{dx} d.\frac{dy}{dx}$, ſumto elemento dx conſtante, erit $q = \frac{ddy}{dx^2}$; ſumto elemento dy conſtante erit $q = -\frac{dyddx}{dx^3}$; nullo autem ſumto conſtante erit $q = \frac{dxddy - dyddx}{dx^3}$ quibus ſimplicitas earum formularum haud mediocriter offuſcatur.

Coroll. 4.

717. Si elementum $\sqrt{(dx^2 + dy^2)}$, quod ſaepe fit, conſtans accipiatur, erit $dxddx + dyddy = 0$; vnde poſtremus valor ipſius q vel ob $ddy = -\frac{dxddx}{dy}$ abit in $q = -\frac{(dx^2 + dy^2)ddx}{dx^3 dy}$ vel ob $ddx = -\frac{dyddy}{dx}$ abit in $q = \frac{(dx^2 + dy^2)ddy}{dx^4}$.

Scho-

Scholion.

718. Repudiata ergo penitus vulgari ratione aequationes differentio - differentiales exprimendi, quippe qua formulae in se satis simplices vehementer complicatae euadere possent, ratione hic stabilita vtamur, indeqne resolutionem huiusmodi formularum simplicium doceamus.

Problema 92.

719. Posito $dy = pdx$ et $dp = qdx$ si q aequetur functioni cuicunque ipsius p, inuenire relationem inter ipsas variabiles x et y.

Solutio.

Sit ergo $q = P$ denotante P functionem quamcunque ipsius p: quoniam igitur est $q = \frac{dp}{dx}$, erit $dp = Pdx$ hincque $dx = \frac{dp}{P}$ et $dy = pdx = \frac{pdp}{P}$. Ex quo consequimur integrando:

$$x = a + \int \frac{dp}{P} \text{ et } y = b + \int \frac{pdp}{P}$$

ita vt tam x, quam y per eandem nouam variabilem p determinetur. Atque cum duae nouae constantes a et b per duplicem integrationem sint introductae hoc integrale pro completo erit habendum.

Coroll. 1.

720. Aequatio $q = P$, cuius integrationem hic tradidimus, si in formam consuetam sumto dx,

constante resoluatur ob $q = \frac{ddy}{dx^2}$, transmutabitur in $ddy = dx^2 f : \frac{dy}{dx}$; quae est aequatio differentio-differentialis, in qua ipsae variabiles x et y non occurrunt.

Coroll. 2.

721. Talis quoque forma prodit, si elementum dy vel alia expressio differentialis, in quam ipsae x et y non ingrediuntur, veluti $\sqrt{(dx^2 + dy^2)}$ pro constante sumatur. Hoc ergo modo omnis aequatio differentio-differentialis in quam ipsae variabiles x et y non ingrediuntur, integrari poterit.

Coroll. 3.

722. Sin autem huiusmodi elementum $ydx - xdy$ constans assumatur, vt $yddx - xddy = 0$, ob

$$q = \frac{1}{dx} d . \frac{dy}{dx} = \frac{dxddy - dyddx}{dx^3}, \text{ fiet}$$

$$q = \frac{(ydx - xdy)ddx}{xdx^3} = \frac{(ydx - xdy)ddy}{ydx^3}$$

quae expressio, si aequetur functioni ipsius $p = \frac{dx}{dy}$, integrari poterit.

Coroll. 4.

723. Si fuerit P quantitas constans, vt sit $q = f$ erit

$$x = a + \frac{p}{f} \text{ et } y = b + \frac{pp}{2f},$$

vnde

vnde fit

$$y = b + \tfrac{f}{2}(x-a)^2 \text{ seu } y = \tfrac{1}{2}fxx - afx + \tfrac{1}{2}aaf + b$$

seu mutatata forma constantium $y = \tfrac{1}{2}fxx + Cx + D$.

Scholion.

724. Cum scilicet aequatio differentio - differentialis duplici integratione indigeat, si vtraque omni extensione instituatur, duae nouae constantes arbitrariae introducuntur; in quo criterium, num huiusmodi integrale sit completum, consistit. Quemadmodum enim aequationum differentialium primi gradus integratio completa vnam constantem arbitrariam implicat, ita si aequatio differentialis fuerit secundi gradus, binae constantes nouae in integrale completum ingredientur, ternae autem ac plures, si aequatio differentialis fuerit tertii altiorisue gradus. Problemata autem, quorum resolutio ad huiusmodi aequationes differentiales altiorum graduum deducunt, natura sua ita sunt comparata, vt solutionis determinatio totidem constantes requirat. Ita in aequatione $q = f$, seu sumto dx constante $ddy = fdx^2$ aequatio integralis completa $y = \tfrac{1}{2}fxx + Cx + D$ duas constantes nouas C et D inuoluit, quod etiam in subiunctis exemplis patebit.

Exemplum 1.

725. *Aequationis differentio-differentialis* $a\,ddy = dx\,dy$ *in qua elementum* dx *constans est sumtum, integrale completum inuenire.*

Posito $dy = p\,dx$ et $dp = q\,dx$, erit $ddy = q\,dx^2$, hincque $aq = p$, et $P = \frac{p}{a}$. Quocirca integratio praebet:

$$x = \int \frac{a\,dp}{p} = C + a\,lp \text{ et } y = \int a\,dp = D + ap.$$

Cum igitur sit

$$p = \frac{y - D}{a}, \text{ erit } x = C + a\,l\frac{y-D}{a}$$

quae est aequatio integralis completa binas constantes C et D inuoluens.

Exemplum 2.

726. *Posito* dx *constante inuenire aequationem inter* x *et* y *vt fiat* $\frac{(dx^2 + dy^2)\sqrt{(dx^2 + dy^2)}}{-dx\,ddy} = a$.

Posito $dy = p\,dx$, ob dx constans erit $ddy = dp\,dx$ sicque nostra aequatio est $\frac{(1+pp)\sqrt{(1+pp)}}{-dp}\,dx = a$, vnde fit

$$dx = \frac{-a\,dp}{(1+pp)^{\frac{3}{2}}} \text{ et } dy = \frac{-ap\,dp}{(1+pp)^{\frac{3}{2}}}.$$

Per integrationem ergo nanciscimur:

$$x = A - \frac{ap}{\sqrt{(1+pp)}} \text{ et } y = B + \frac{a}{\sqrt{(1+pp)}}$$

vnde

vnde concludimus

$$(A-x)^2+(y-B)^2=aa.$$

Corollarium.

727. Si x et y denotent coordinatas rectangulas lineae curuae, formula $\frac{(dx^2+dy^2)\sqrt{(dx^2+dy^2)}}{-dxddy}$ exprimit eius radium osculi, qui ergo vt sit constans $=a$, aequatio integralis inuenta circulum radio a describendum indicat.

Exemplum 3.

728. *Posito* $ds=\sqrt{(dx^2+dy^2)}$ *eoque sumto constante inuenire aequationem inter* x *et* y, *vt fiat* $\frac{dsdy}{ddx}=\frac{adx}{dy}$.

Ponatur $dy=pdx$, erit $ds=dx\sqrt{(1+pp)}$, et ob ds constans

$$ddx\sqrt{(1+pp)}+\frac{pdxdp}{\sqrt{(1+pp)}}=0 \text{ seu } ddx=\frac{-pdxdp}{1+pp},$$

vnde aequatio proposita abit in

$$\frac{pdx^2\sqrt{(1+pp)}}{-pdxdp}(1+pp)=\frac{a}{p} \text{ seu } dx=\frac{-adp}{p(1+pp)^{\frac{3}{2}}}$$

$$\text{et } dy=\frac{-adp}{(1+pp)^{\frac{3}{2}}} \text{ ergo } y=D-\frac{ap}{\sqrt{(1+pp)}}.$$

At pro illa formula statuatur $p=\frac{1}{r}$, eritque

$$dx=\frac{arrdr}{(rr+1)^{\frac{3}{2}}}=\frac{adr}{\sqrt{(rr+1)}}-\frac{adr}{(rr+1)^{\frac{3}{2}}}$$

 vnde

vnde fit integrando:

$$x = C - \frac{ar}{\sqrt{(1+rr)}} + a\,l(r + \sqrt{(1+rr)}) \text{ seu}$$
$$x = C - \frac{a}{\sqrt{(1+pp)}} + a\,l\frac{1+\sqrt{(1+pp)}}{p}.$$

Exemplum 4.

729. *Posito* $ds = \sqrt{(dx^2 + dy^2)}$ *hocque elemento sumto constante, fieri oporteat* $\frac{ds\,dy}{d\,dx} = a \text{ Ang. tang } \frac{dy}{dx}$.

Si fiat vt ante $dy = p\,dx$, orietur haec aequatio integranda:

$$\frac{-dx(1+pp)^{\frac{3}{2}}}{dp} = a \text{ Ang. tang.} p \text{ seu}$$

$$dx = \frac{-adp}{(1+pp)^{\frac{3}{2}}} \text{ Ang. tang.} p \text{ et } dy = \frac{-apdp}{(1+pp)^{\frac{3}{2}}} \text{ Ang. tang.} p.$$

Cum nunc sit diff. Ang. tang. $p = \frac{dp}{1+pp}$ erit

$$x = \frac{-ap}{\sqrt{(1+pp)}} \text{ Ang. tang.} p + a\int \frac{pdp}{(1+pp)^{\frac{3}{2}}} \text{ et}$$

$$y = \frac{a}{\sqrt{(1+pp)}} \text{ Ang. tang.} p - a\int \frac{dp}{(1+pp)^{\frac{3}{2}}}.$$

Quamobrem colligemus:

$$x = C - \frac{ap}{\sqrt{(1+pp)}} - \frac{ap}{\sqrt{(1+pp)}} \text{ Ang. tang.} p \text{ et}$$
$$y = D - \frac{a}{\sqrt{(1+pp)}} + \frac{a}{\sqrt{(1+pp)}} \text{ Ang. tang.} p.$$

Coroll. 1.

Coroll. 1.

730. Si x ſit abſciſſa et y applicata curuae, radius oſculi proportionalis eſſe debet angulo, quem curuae tangens cum axe conſtituit; vnde patet hanc curuam fore quandam ſpiralem, circa originem abſciſſarum ſe euoluentem.

Coroll. 2.

731. Si angulus ille cuius tangens $= p$, ponatur $= \Phi$, erit $p = \text{tang.}\,\Phi$, hincque

$$x = C - a\cos.\Phi - a\Phi\sin.\Phi \text{ et}$$

$$y = D - a\sin.\Phi + a\Phi\cos.\Phi$$

vnde colligitur

$$x\cos.\Phi + y\sin.\Phi = C\cos.\Phi + D\sin.\Phi - a.$$

Coroll. 3.

732. Vt ſumto $\Phi = 0$, ambae x et y euaneſcant, ſumi debet $C = a$ et $D = 0$, eritque

$$x = a - a\cos.\Phi - a\Phi\sin.\Phi \text{ et } y = -a\sin.\Phi + a\Phi\cos.\Phi$$

vnde quamdiu angulus Φ eſt minimus, erit

$$x = -\tfrac{1}{2}a\Phi\Phi + \tfrac{1}{8}a\Phi^4 \text{ et } y = -\tfrac{1}{3}a\Phi^3 + \tfrac{1}{30}\Phi^5$$

ideoque proxime

$$\frac{x^3}{yy} = -\tfrac{9}{8}a \text{ ſeu } yy = -\frac{8x^3}{9a}.$$

Proble-

Problema 93.

733. Posito $dy = pdx$ et $dp = qdx$, si quantitas q aequetur functioni ipsius x, quae sit X, definire relationem inter binas variabiles x et y.

Solutio.

Cum ergo sit $q = X$ erit $qdx = dp = Xdx$, vnde integrando colligimus $p = \int Xdx + C$, atque hinc ob $dy = pdx$ adipiscemur

$$y = \int dx \int X dx + Cx + D.$$

At est

$$\int dx \int Xdx = x\int Xdx - \int Xxdx$$

vti sumendis differentialibus sponte patet. Quare aequatio integralis completa relationem inter binas variabiles x et y continens est

$$y = x\int Xdx - \int Xxdx + Cx + D$$

duas constantes arbitrarias C et D inuoluens. Quae ergo erit algebraica, si ambae formulae differentiales Xdx et $Xxdx$ integrationem admittant.

Coroll. 1.

734. Quodsi ergo sit $q = 0$, seu sumto dx constante $ddy = 0$, vt sit $X = 0$, erit aequatio integralis completa $y = Cx + D$.

Coroll. 2.

Coroll. 2.

735. Aequationes ergo differentio - differentiales, quas hoc modo integrare licet, sumto dx constante, in hac forma $ddy = Xdx^2$ continentur, vnde prima integratio praebet $dy = dx \int X dx + C$ et altera $y = \int dx \int X dx + Cx + D$.

Coroll. 3.

736. Sin autem differentiale dy capiatur constans ob $p = \frac{dy}{dx}$ erit $dp = -\frac{dy\,ddx}{dx^2} = q\,dx$ forma aequationum hoc modo integrandarum erit $-dyddx = Xdx^3$.

Coroll. 4.

737. Quodsi elementum $ds = \sqrt{(dx^2 + dy^2)}$ sit constans ob $dxddx + dyddy = 0$ erit $dp = \frac{dxddy - dyddx}{dx^2} = -\frac{ds^2\,ddx}{dx^2\,dy} = q\,dx$. Hinc forma aequationum hoc modo integrandarum est $-ds^2\,ddx = X dx^3\,dy$. Vel cum etiam sit $dp = q\,dx = +\frac{ds^2\,ddy}{dx^3}$, ea erit $ds^2\,ddy = X\,dx^4$.

Scholion.

738. Hic manifestum est, quantum intersit aequationes differentio - differentiales a forma solita, vbi elementum quodpiam constans est assumtum, ab hac conditione liberare et ad formam hic stabilitam reducere. Si enim proponatur haec aequatio $ds^2 ddy = Xdx^4$,

in qua elementum $ds = \sqrt{(dx^2 + dy^2)}$ conſtans ſit aſſumtum, haud facile patet, quomodo eius integratio ſit ſuſcipienda. Noſtra autem methodo, ſi ponamus $dy = pdx$, vt ſit $ds = dx\sqrt{(1+pp)}$ et $ddy = pddx + dxdp$, induit iſta aequatio hanc formam:

$$dx^2(1+pp)(pddx+dxdp) = Xdx^4 \text{ ſeu}$$
$$(pddx+dxdp)(1+pp) = Xdx^2.$$

At quia ds ac proinde quoque $ds^2 = dx^2(1+pp)$ eſt conſtans, erit

$$ddx(1+pp)+pdxdp = 0 \text{ ſeu } ddx = \frac{-pdxdp}{1+pp},$$

ideoque $pddx + dxdp = \frac{dxdp}{1+pp}$,

ita vt fiat $dp = Xdx$, quae aequatio iam facillime tractatur. Hic ſcilicet in ſubſidium vocari debent ea, quae ſupra de integratione formularum differentialium ſimplicium ſunt tradita.

Exemplum 1.

739. *Sumto* dx *conſtante, ſi fuerit* $ddy = ax^n dx^2$, *integrale completum inueſtigare.*

Cum ſit $\frac{ddy}{dx} = ax^n dx$, ob dx conſtans erit integrando $\frac{dy}{dx} = \frac{a}{n+1}x^{n+1} + C$, hincque denuo integrando:

$$y = \frac{a}{(n+1)(n+2)}x^{n+2} + Cx + D$$

vbi caſus $n = -1$ et $n = -2$ ſeorſim ſunt euoluendi.

I.

I. Ergo si $n = -1$ erit $\frac{ddy}{dx} = \frac{\alpha dx}{x}$, hincque $\frac{dy}{dx} = \alpha lx + C$, vnde cum sit $dy = \alpha dx lx + C dx$, erit denuo integrando $y = \alpha x lx - \alpha x + Cx + D$, seu loco $C - \alpha$ scribendo C habebitur $y = \alpha x lx + Cx + D$.

II. Si $n = -2$ et $\frac{ddy}{dx} = \frac{\alpha dx}{x^2}$ erit $\frac{dy}{dx} = \frac{-\alpha}{x} + C$ hincque $y = -\alpha lx + Cx + D$.

Exemplum 2.

740. *Posito* $ds = \sqrt{(dx^2 + dy^2)}$ *constante, si fuerit* $\frac{ds^2 ddy}{dx^4} = \frac{1}{a}\cos.\frac{x}{c}$, *inuenire integrale completum.*

Ex superioribus constat fore $\frac{ds^2 ddy}{dx^4} = q$, ita vt proposita sit haec aequatio $q = \frac{1}{a}\cos.\frac{x}{c}$, vnde fit

$$qdx = dp = \frac{dx}{a}\cos.\frac{x}{c}$$

et integrando

$$p = \frac{c}{a}\sin.\frac{x}{c} + C = \frac{dy}{dx}.$$

Quare obtinebitur

$$y = -\frac{cc}{a}\cos.\frac{x}{c} + Cx + D,$$

quae est aequatio integralis completa.

Problema 94.

741. Posito $dy = pdx$ et $dp = qdx$, si quantitas q aequetur functioni cuicunque ipsius y tantum, quae sit Y, inuenire aequationem integralem completam inter x et y.

 Solutio.

Solutio.

Cum fit $q = Y = \frac{dp}{dx}$ erit $dx = \frac{dp}{Y}$, hincque $pdx = dy = \frac{pdp}{Y}$, vnde conficitur haec aequatio inter p et y feparata $pdp = Ydy$, quae integrata praebet:

$$\tfrac{1}{2}pp = \int Y dy + \tfrac{1}{2}C \text{ et } p = \sqrt{(C + 2\int Y dy)} = \frac{dy}{dx}.$$

Hinc ergo porro concluditur $x = \int \frac{dy}{\sqrt{(C + 2\int Y dy)}}$ quae integratio denuo conftantem arbitrariam inducit, ita vt hoc modo aequatio integralis completa inter x et y obtineatur.

Coroll. 1.

742. Cum fit $q = \frac{dp}{dx}$ et $dx = \frac{dy}{p}$ erit $q = \frac{pdp}{dy}$. Quare cum aequatio propofita fit $q = Y$, erit $\frac{pdp}{dy} = Y$, hincque $pdp = Ydy$, vnde praecedens integratio fponte deducitur.

Coroll. 2.

743. Sumto elemento dx conftante, cum fit $q = \frac{ddy}{dx^2}$ aequationes hic integratae habebunt hanc formam $ddy = Ydx^2$, cuius integratio fi per dy multiplicetur, eft manifefta, fit enim

$$\tfrac{1}{2}dy^2 = dx^2 \int Y dy + \tfrac{1}{2}C dx^2$$

ob dx conftans, hincque $dx = \frac{dy}{\sqrt{(C + 2\int Y dy)}}$ vt ante.

Scholion.

Scholion.

744. En ergo specimen aequationum differentialium, quae per idoneum multiplicatorem integrabiles redduntur, ex quo intelligitur hanc methodum etiam in his aequationibus vsum habere posse; deinceps autem locus erit hanc methodum vberius excolendi, cuius quippe vsus praecipue in aequationibus differentialibus altiorum graduum est insignis, vbi variabilium separatio nihil subsidii affert. Atque hanc ob causam iam supra hanc methodum per multiplicatores integrandi commendauimus, alterique per separationem praecedenti longe antetulimus.

Exemplum 1.

745. *Posito* dx *constante, si fuerit*

$$aaddy = ydx^2$$

inuenire integrale completum.

Multiplicetur aequatio proposita per $2dy$, vt prodeat

$$2aadyddy = 2ydydx^2,$$

quae ob dx constans habebit integrale:

$$aady^2 = yydx^2 + Cdx^2,$$

vnde colligitur:

$$dx = \frac{ady}{\sqrt{(yy+C)}},$$

 quae

quae denuo integrata dat

$$x = al(y + \surd(yy + C)) - alb$$

vnde concludimus sumto e pro numero cuius logarithmus est $= 1$, fore

$$be^{\frac{x}{a}} = y + \surd(yy + C)$$

et irrationalitatem tollendo

$$bbe^{\frac{2x}{a}} - 2bye^{\frac{x}{a}} = C$$

ita vt sit

$$y = \tfrac{1}{2}be^{\frac{x}{a}} - \tfrac{C}{2b}e^{-\frac{x}{a}}$$

at forma constantium C et b mutata habebitur:

$$y = Ce^{\frac{x}{a}} + De^{-\frac{x}{a}}$$

quae est aequatio integralis completa.

Exemplum 2.

746. *Posito* dx *constante si fuerit*

$$aaddy + ydx^2 = 0,$$

inuenire integrale completum.

Multiplicatione per $2dy$ facta aequationis

$$2aadyddy + 2ydydx^2 = 0$$

integrale est

$$aady^2 + yydx^2 = ccdx^2$$

vnde

vnde deducimus:

$$dx = \frac{a\,dy}{\sqrt{(cc-yy)}},$$

quae denuo integrata dat

$$x = a\,\text{Ang. fin.}\frac{y}{c} + b.$$

Erit ergo

$$\frac{y}{c} = \text{fin.}\frac{x-b}{a} = \text{cof.}\frac{b}{a}\,\text{fin.}\frac{x}{a} - \text{fin.}\frac{b}{a}\,\text{cof.}\frac{x}{a}$$

vel mutatis conſtantibus b et c ita vt ſit

$$c\,\text{cof.}\frac{b}{a} = C \text{ et } -c\,\text{fin.}\frac{b}{a} = D \text{ erit}$$

$$y = C\,\text{fin.}\frac{x}{a} + D\,\text{cof.}\frac{x}{a}.$$

Vel retenta prima forma habemus

$$y = C\,\text{fin.}\left(\frac{x}{a} + \alpha\right).$$

Corollarium.

747. Hoc exemplum ex praecedente reſolui potuiſſet, cum ſit

$e^{u\sqrt{-1}} = \text{cof.}u + \sqrt{-1}.\text{fin.}u$ et $e^{-u\sqrt{-1}} = \text{cof.}u - \sqrt{-1}.\text{fin.}u$,

ac viciſſim

$\text{cof.}u\sqrt{-1} = \frac{1}{2}e^{u} + \frac{1}{2}e^{-u}$ et $\text{fin.}u\sqrt{-1} = \frac{1}{2\sqrt{-1}}e^{u} - \frac{1}{2\sqrt{-1}}e^{-u}$.

Exemplum 3.

748. *Poſito* dx *conſtante ſi fuerit*

$$ddy\sqrt{ay} = dx^2,$$

integrale completum inuenire.

Cum

Cum ergo sit $2dyddy = \frac{2dy}{\sqrt{ay}} . dx^2$

erit integrando

$$dy^2 = \frac{4dx^2\sqrt{y}}{\sqrt{a}} + 4ndx^2 = \frac{4dx^2(\sqrt{y}+n\sqrt{a})}{\sqrt{a}}$$

vnde colligimus

$$2dx = \frac{dy\sqrt[4]{a}}{\sqrt{(\sqrt{y}+n\sqrt{a})}}.$$

Sit commoditatis gratia $n\sqrt{a} = b$ et $\sqrt{y} = z$,

vt fiat $dy = 2zdz$ et $\frac{dx\sqrt{n}}{\sqrt{b}} = \frac{zdz}{\sqrt{(b+z)}}$,

cuius integrale est:

$$\frac{x\sqrt{n}}{\sqrt{b}} = \tfrac{2}{3}(z-2b)\sqrt{(b+z)} + C \text{ seu restituendo}$$

$$\frac{x}{\sqrt[4]{a}} = \tfrac{2}{3}(\sqrt{y}-2\sqrt{c})\sqrt{(\sqrt{y}+\sqrt{c})} + C$$

vbi c et C sunt binae constantes arbitrariae. Erit ergo

$$\frac{3(x+f)}{2\sqrt[4]{a}} = (\sqrt{y}-2\sqrt{c})\sqrt{(\sqrt{y}+\sqrt{c})}$$

posito $C = \frac{-f}{\sqrt[4]{a}}$, et sumtis quadratis

$$\frac{9(x+f)^2}{4\sqrt{a}} = y\sqrt{y} - 3y\sqrt{c} + 4c\sqrt{c}.$$

Scholion.

Scholion.

749. Forma ergo vulgaris aequationum hoc modo integrandarum ſumto elemento dx conſtante, eſt $ddy = Y dx^2$, quae per dy multiplicata manifeſto fit integrabilis. Sin autem elementum dy capiatur conſtans ob $q = \frac{dp}{dx}$ et $p = \frac{dy}{dx}$ erit $q = -\frac{dy\,ddx}{dx^3}$, hincque forma vulgaris $dy\,ddx = -Y dx^3$. Porro ſumto elemento $ds = V(dx^2 + dy^2)$ conſtante, vt ſit $dx\,ddx + dy\,ddy = 0$, ob $dp = \frac{dx\,ddy - dy\,ddx}{dx^2}$ erit vel $q = -\frac{ds^2\,ddx}{dx^3\,dy}$ vel $q = \frac{ds^2\,ddy}{dx^4}$, vnde naſcitur haec forma

$$-\frac{ds^2\,ddx}{dx^3\,dy} = Y \text{ vel } \frac{ds^2\,ddy}{dx^4} = Y$$

quae etiam per dy multiplicatae integrabiles euadunt, etiamſi hoc iam minus pateat. Simili modo ſi elementum ydx ſumatur conſtans, vt ſit $yddx + dxdy = 0$ et $ddx = -\frac{dx\,dy}{y}$, ob $dp = \frac{ddy}{dx} + \frac{dy^2}{y\,dx}$, orietur haec forma $yddy + dy^2 = Y y dx^2$; cuius membrum prius integrabile redditur, ſi per functionem quamcunque ipſarum ydy et ydx multiplicetur, ergo etiam per $\frac{y\,dy}{yy\,dx^2}$, quo multiplicatore ſimul alterum membrum $Y y dx^2$ redditur integrabile. His igitur caſibus ſimpliciſſimis aequationum differentio - differentialium expeditis, qui ne vlla quidem difficultate laborant, ad difficiliores progrediamur, ac primo quidem ad eas aequationes, in quibus altera binarum variabi-

lium x et y ipſa non ineſt; ita vt aequatio propoſita ternas tantum contineat litteras x, p et q vel y, p et q, vtriusque enim ratio fere perinde eſt comparata.

CAPVT II.

CAPVT II.

DE AEQVATIONIBVS DIFFERENTIO - DIFFERENTIALIBVS IN QVIBVS ALTERA VARIABILIVM IPSA DEEST.

Problema 95.

750.

Posito $dy = pdx$ et $dp = qdx$, si detur aequatio quaecunque inter tres quantitates x, p et q, in quam altera variabilis y non ingrediatur, inuestigare relationem inter ipsas variabiles x et y.

Solutio.

Cum aequatio proposita has tres quantitates x, p et q contineat loco q scribatur eius valor $\frac{dp}{dx}$, atque habebitur aequatio differentialis primi gradus duas tantum quantitates variabiles x et p inuoluens, quam secundum praecepta prioris partis tractari, eiusque integrale inuestigari oportet. Integrali autem inuento, quod si fuerit completum constantem arbitrariam complectetur, inde vel p per x, vel x per p determinari poterit. Priori casu, quo p per x

definire licet, vt p aequetur functioni cuidam ipsius x, quae sit $= X$, ob $p = X$ fiet $pdx = dy = Xdx$, vnde reperitur $y = \int Xdx +$ Const. quae aequatio relationem desideratam inter x et y definit. Posteriori casu quo x per p datur, et functioni cuidam P ipsius p aequatur, vt sit $x = P$ erit $y = \int pdx = \int pdP$ seu $y = Pp - \int Pdp$.

Sin autem neque x per p neque p, per x definiri queat, videndum est, num vtramque per nouam variabilem u exprimere liceat, vnde fiat $x = V$ et $p = U$; tum enim habebitur $y = \int UdV$.

Coroll. 1.

751. Huiusmodi ergo aequationum differentio-differentialium resolutio ita instituitur, vt reuocetur ad aequationem differentialem primi gradus inter binas variabiles x et p; quae si integrari queat, simul illius aequationis integratio habebitur, accedente quadam noua integratione.

Coroll. 2.

752. Si aequatio inter x, p et q proposita ita fuerit comparata, vt q vnicam dimensionem non excedat, vel si ad talem formam reduci patiatur, orietur aequatio differentialis simplex, differentialia vnius tantum dimensionis inuoluens, vbi praecepta ante tradita in vsum sunt vocanda.

Coroll. 3.

Coroll. 3.

753. Sin autem quantitas q plures obtineat dimenſiones vel adeo tranſcendenter ingrediatur, tentanda ſunt ea artificia, quae in fine ſuperioris partis circa reſolutionem huiusmodi aequationum ſunt tradita.

Scholion.

754. Quando in aequatione inter x, p et q littera q vnicam habet dimenſionem, indeque poſito $q = \frac{dp}{dx}$ aequatio differentialis ſimplex naſcitur, praecipui caſus, quibus integratio ſuccedit, ſunt: 1) ſi aequatio haec differentialis ſeparationem admittat, 2) ſi alterutra variabilium p et x, differentialium quoque ratione habita, vnam dimenſionem non ſuperet; ac 3) ſi ambae variabiles x et p vbique eandem dimenſionum numerum conſtituant, quo caſu aequatio homogenea appellatur. Caſus minus late patentes, cuiusmodi ſupra euoluimus, hic non commemoramus. Deinde ſi quantitas q vel pluribus dimenſionibus ſit implicata, vel adeo tranſcendenter ingrediatur, caſus praecipui reſolutionem admittentes, quemadmodum ſupra docuimus, ſunt: 1) Si proponatur aequatio quaecunque inter x et q deficiente p. 2) Si aequatio tantum p et q contineat, quos binos quidem caſus iam capite praecedente tractauimus. 3) Si in aequatione propoſita binae variabiles p et x vbique eundem dimenſionum nume-

rum conſtituant. 4) Si in aequatione inter x, p et q altera binarum litterarum x vel p vnicam dimenſionem obtineat. 5) Si aequatio ita fuerit comparata, vt poſito $x = v^{\mu}$; $p = z^{\mu + \nu}$ et $q = t^{\nu}$ aequatio oriatur homogenea inter v, z et t, quae ſcilicet vbique eundem dimenſionum numerum conſtituant. Secundum hos ergo caſus exempla proferamus

Exemplum I.

755. *Inueſtigare aequationem inter* x *et* y, *vt poſito* dx *conſtante haec formula* $\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\,ddy}$ *aequetur datae functioni ipſius* x, *quae ſit* $= X$.

Poſito $dy = p\,dx$ et $dp = q\,dx$, erit

$$\frac{(1+pp)^{\frac{3}{2}}}{q} = X = \frac{(1+pp)^{\frac{3}{2}}dx}{dp} \text{ ideoque}$$

$$\frac{dx}{X} = \frac{dp}{(1+pp)^{\frac{3}{2}}},$$

vbi cum variabiles x et p ſint a ſe inuicem ſeparatae, integratio dat:

$$\frac{p}{\sqrt{(1+pp)}} = \int \frac{dx}{X}.$$

Ponatur $\int \frac{dx}{X} = V$ integrali completo ſumto, erit V functio ipſius x, hinc

$$p = V\sqrt{(1+pp)} \text{ et } p = \frac{V}{\sqrt{(1-VV)}}.$$

Quare

Quare $dy = p\,dx = \frac{V\,dx}{\sqrt{(1-VV)}}$,

vnde obtinetur

$$y = \int \frac{V\,dx}{\sqrt{(1-VV)}}.$$

Tum vero praeterea elicitur elementum

$$\sqrt{(dx^2+dy^2)} = dx\sqrt{(1+pp)} = \frac{dx}{\sqrt{(1-VV)}},$$

cuius integrale praebet

$$\int dx\sqrt{(1+pp)} = \int \frac{dx}{\sqrt{(1-VV)}}.$$

Coroll. 1.

756. Si x et y ſint coordinatae orthogonales curvae, erit formula $\frac{(1+pp)^{\frac{3}{2}}}{q}$ eius radius curuedinis, vnde hinc curua definitur, cuius radius curuedinis aequetur functioni cuicunque abſciſſae x.

Coroll. 2.

757. Si ergo radius curuedinis debeat eſſe reciproce proportionalis abſciſſae x, ſumatur $X = \frac{aa}{2x}$, eritque

$$V = \int \frac{2x\,dx}{aa} = \frac{xx+ab}{aa}, \text{ hinc}$$

$$y = \int \frac{(xx+ab)\,dx}{\sqrt{(a^4-(xx+ab)^2)}},$$

quae conditio praebet curuas a lamina elaſtica formatas.

Coroll. 3.

Coroll. 3.

758. Si fit $V = x^n$, feu $X = \frac{1}{nx^{n-1}}$, neglecta conftante addenda, oritur $y = \int \frac{x^n dx}{\sqrt{(1 - x^{2n})}}$, quod integrale algebraice exhiberi poteft cafibus, quibus eft vel $n = \frac{1}{2i+1}$ vel $n = \frac{-1}{2i}$ denotante i numerum integrum pofitiuum.

Exemplum 2.

759. *Si pofito* dx *conftante oporteat effe*

$$dx(dx^2 + dy^2) + xdyddy = addy\sqrt{(dx^2 + dy^2)}$$

inuenire aequationem inter **x** *et* **y**.

Pofito $dy = pdx$, noftra aequatio ob $ddy = dpdx$ induit hanc formam

$$dx(1 + pp) + xpdp = adp\sqrt{(1 + pp)},$$

quae per $\sqrt{(1 + pp)}$ diuifa fit integrabilis, oritur enim:

$$x\sqrt{(1 + pp)} = ap + b \text{ feu } x = \frac{ap + b}{\sqrt{(1 + pp)}}.$$

Cum nunc fit

$$y = \int pdx = px - \int xdp \text{ erit}$$

$$y = \frac{app + bp}{\sqrt{(1 + pp)}} - \int \frac{dp(ap + b)}{\sqrt{(1 + pp)}}$$

et integratione euoluta:

$$y = \frac{app + bp}{\sqrt{(1 + pp)}} - a\sqrt{(1 + pp)} - bl\frac{p + \sqrt{(1 + pp)}}{n}$$

feu

seu $y = \frac{bp - a}{\sqrt{(1 + pp)}} - bl\frac{p + \sqrt{(1 + pp)}}{n}$

ita vt ambae variabiles x et y per p definiantur. Cum igitur ex priori eliciatur:

$p = \frac{ab + x\sqrt{(aa + bb - xx)}}{xx - aa}$ et $\sqrt{(1 + pp)} = \frac{bx + a\sqrt{(aa + bb - xx)}}{xx - aa}$

erit his valoribus substitutis:

$$y = \frac{a(aa + bb - xx) + bx\sqrt{(aa + bb - xx)}}{bx + a\sqrt{(aa + bb - xx)}} - bl\frac{b + \sqrt{(aa + bb - xx)}}{n(x - a)}$$

seu $y = \sqrt{(aa + bb - xx)} - bl\frac{b + \sqrt{(aa + bb - xx)}}{n(x - a)}$.

Corollarium.

760. Si constans priori integratione ingressa b euanescens sumatur, aequatio inter x et y fit algebraica, erit enim $y = \sqrt{(aa - xx)}$. Sin autem b non euanescat, aequatio integralis est transcendens, et logarithmos inuoluit.

Exemplum 3.

761. *Posito* dx *constante si debeat esse:*

$$aaddy\sqrt{(aa + xx)} + aadxdy = xxdx^2$$

inuenire aequationem inter x *et* y.

Posito $dy = pdx$ habebimus hanc aequationem

$aadp\sqrt{(aa + xx)} + aapdx = xxdx$ seu

$dp + \frac{pdx}{\sqrt{(aa + xx)}} = \frac{xxdx}{aa\sqrt{(aa + xx)}}$

in qua variabilis p vnam dimenfionem non fuperat. Cum ergo fit

$$\int \frac{dx}{\sqrt{(aa+xx)}} = l(x+\sqrt{(aa+xx)}),$$

haec aequatio integrabilis redditur, fi multiplicetur per $x+\sqrt{(aa+xx)}$, tum enim prodit:

$$p(x+\sqrt{(aa+xx)}) = \int \frac{xxdx(x+\sqrt{(aa+xx)})}{aa\sqrt{(aa+xx)}} \text{ feu}$$

$$p(x+\sqrt{(aa+xx)}) = \frac{1}{aa}\int \frac{x^3dx}{\sqrt{(aa+xx)}} + \frac{x^3}{3aa} \text{ at}$$

$$\int \frac{x^3dx}{\sqrt{(aa+xx)}} = \tfrac{1}{3}(xx-2aa)\sqrt{(aa+xx)}+C \text{ hinc}$$

$$p(x+\sqrt{(aa+xx)}) = \frac{(xx-2aa)\sqrt{(aa+xx)}+x^3}{3aa}+C.$$

Haec multiplicetur per $\sqrt{(aa+xx)}-x$ vt prodeat

$$aap = \frac{-xx-2aa+2x\sqrt{(aa+xx)}}{3}+C\sqrt{(aa+xx)}-Cx$$

et quia $dy=pdx$ erit integrando

$$aay = -\tfrac{1}{9}x^3-\tfrac{2}{3}aax+\tfrac{2}{9}(aa+xx)\sqrt{(aa+xx)}$$
$$-\tfrac{1}{2}Cxx+C\int dx\sqrt{(aa+xx)}.$$

Quodfi ergo conftans C euanefcat, aequatio inter x et y erit algebraica; fcilicet

$$9aay+6aax+x^3 = 2(aa+xx)\sqrt{(aa+xx)}.$$

Exemplum 4.

762. *Pofito* dx *conftante inuenire integrale huius aequationis differentio-differentialis*

$$(aady^2+xxdx^2)ddy = nxdx^2dy.$$

Fiat

Fiat $dy = pdx$ et ob $ddy = dpdx$ habebimus:

$$(aapp + xx)dp = npxdx$$

quae aequatio cum ſit homogenea, ſtatuamus $x = pu$, eritque

$$pp(aa+uu)dp = nppu(pdu + udp) \text{ ſeu}$$

$$\frac{dp}{p} = \frac{nudu}{aa + (1-n)uu}$$

quae integrata dat

$$lp = \frac{n}{2(1-n)} l(aa + (1-n)uu) + \text{Conſt.}$$

Hinc colligitur

$$p = C(aa + (1-n)uu)^{\frac{n}{2(1-n)}}, \text{ atque}$$

$$x = Cu(aa + (1-n)uu)^{\frac{n}{2(1-n)}}.$$

Cum nunc ſit

$$y = px - \int xdp \text{ et } dp = Cnudu(aa + (1-n)uu)^{\frac{3n-2}{2(1-n)}}$$

erit

$$y = CCu(aa + (1-n)uu)^{\frac{n}{1-n}} - nCC\int uudu(aa + (1-n)uu)^{\frac{2n-1}{1-n}}.$$

Caſu autem $n = 1$ erit

$$lp = \frac{uu}{2aa} + C \text{ et } u = a\sqrt{2l\tfrac{p}{c}}, \text{ hinc}$$

$$x = ap\sqrt{2l\tfrac{p}{c}} \text{ et } y = app\sqrt{2l\tfrac{p}{c}} - a\int pdp\sqrt{2l\tfrac{p}{c}}.$$

Corollarium.

763. Si fuerit $n = \frac{1}{2}$ erit

$$x = Cu\sqrt{(aa + \tfrac{1}{2}uu)} \text{ et}$$

$$y = CCu(aa + \tfrac{1}{2}uu) - \frac{CCu^5}{6} + D = CCu(aa + \tfrac{1}{3}uu) + D$$

ſicque relatio inter x et y algebraice exprimitur, quod etiam fit ſi $n = \frac{2}{3}$, vel $n = \frac{3}{4}$ vel $n = \frac{4}{5}$ etc.

Exemplum 5.

764. *Poſito* dx *conſtante integrare hanc aequationem differentio-differentialem:*

$$adxdy^2 + xxdxddy = nxdy\sqrt{(dx^4 + aaddy^2)}.$$

Fiat $dy = pdx$ et $dp = qdx$, vt ſit $ddy = qdx^2$, et noſtra aequatio induet hanc formam:

$$app + qxx = npx\sqrt{(1 + aaqq)}$$

quae eſt homogenea inter p et x. Statuatur ergo $p = ux$, fietque

$$auu + q = nu\sqrt{(1 + aaqq)}.$$

Iam vero eſt

$$dp = qdx = udx + xdu,$$

vnde fit $\frac{dx}{x} = \frac{du}{q-u}$.

At ex illa aequatione inter q et u colligitur:

$$q = \frac{auu + nu\sqrt{(1 - nnaauu + a^4u^4)}}{nnaauu - 1} \text{ et}$$

$$q - u = \frac{u(1 + au - nnaauu) + nu\sqrt{(1 - nnaauu + a^4u^4)}}{nnaauu - 1}$$

ſicque

ſicque

$$\frac{dx}{x} = \frac{du}{u} \cdot \frac{nnaauu - 1}{1 + au - nnaauu + n\sqrt{(1 - nnaauu + a^4u^4)}}.$$

Dabitur ergo x per u, hincque etiam $p = ux$ per u; vnde deducitur $y = \int p\,dx = \int ux\,dx$.

Coroll. 1.

765. Illa aequatio differentialis transformatur in hanc

$$\frac{dx}{x} = \frac{du}{u} \cdot \frac{1 + au - nnaauu - n\sqrt{(1 - nnaauu + a^4u^4)}}{nn - 1 - 2au + (nn - 1)aauu}$$

vnde ratio integrationis facilius perſpicitur.

Coroll. 2.

766. Notatu dignus autem eſt caſus $nn = 2$, quo fit

$$\frac{dx}{x} = \frac{du}{u} \cdot \frac{1 + au - 2aauu - (1 - aauu)\sqrt{2}}{(1 - au)^2} \text{ ſeu}$$

$$\frac{dx}{x} = \frac{du}{u} \cdot \frac{1 + 2au - (1 + au)\sqrt{2}}{1 - au} = \frac{du(1 - \sqrt{2})}{u} + \frac{adu(3 - 2\sqrt{2})}{1 - au}$$

vnde colligitur

$$lx = (1 - \sqrt{2})lu - (3 - 2\sqrt{2})l(1 - au) + \text{Conſt.}$$

ſeu $xu^{\sqrt{2}-1}(1 - au)^{3-2\sqrt{2}} = C$.

Exemplum 6.

767. *Sumto elemento* $ds = \sqrt{(dx^2 + dy^2)}$ *conſtante inuenire integrale huius aequationis*

$$dx^2dy - xds^2ddy = adxds\sqrt{(ddx^2 + ddy^2)}.$$

 Poſito

Posito $dy=pdx$ erit $ds=dx\sqrt{(1+pp)}$, et ob $dds=0$ fit

$$ddx=-\frac{pdpdx}{1-pp}=\frac{-pqdx^2}{1+pp},$$

existente $dp=qdx$, tum vero

$$ddy=pddx+dpdx=\frac{-ppqdx^2}{1+pp}+qdx^2=\frac{qdx^2}{1+pp},$$

ideoque $\sqrt{(ddx^2+ddy^2)}=\frac{qdx^2}{\sqrt{(1+pp)}}$,

quibus substitutis aequatio nostra induit hanc formam:

$p-qx=aq$, quae differentiata praebet $-xdq=adq$

ideoque $dq=0$ et $q=\frac{1}{c}$. Hinc $p=\int qdx=\frac{x+a}{c}$, qui idem valor ex aequatione $p=(x+a)q$ sine integratione obtinetur. Tum vero est $y=\int pdx=\frac{xx+2ax}{2c}+b$, quae est aequatio integralis completa binas constantes b et c inuoluens.

Exemplum 7.

768. *Sumto elemento* $ds=\sqrt{(dx^2+dy^2)}$ *constante, inuenire integrale huius aequationis differentio-differentialis:*

$$dx^2dy-xds^2ddy=\frac{bdx^4ds^2ddy}{\sqrt{(dx^4+aads^4ddy^2)}}.$$

Posito $dy=pdx$ et $dp=qdx$ ob $ds=dx\sqrt{(1+pp)}$, erit

$$ddx=\frac{-pqdx^2}{1+pp} \text{ et } ddy=\frac{-dxddx}{dy}=\frac{-ddx}{p}=\frac{qdx^2}{1+pp}$$

ergo $ds^2ddy=qdx^4$, vnde aequatio nostra fit

$$p-qx=\frac{bq}{\sqrt{(1+aaqq)}}$$

quae

quae differentiata praebet:

$$-x\,dq = \frac{b\,dq}{\sqrt{(1+aaqq)^{3}}}$$

vnde concluditur vel $dq = 0$ vel $x = \frac{-b}{(1+aaqq)^{\frac{3}{2}}}$.

Priori casu est $q = \frac{1}{c}$, et $p = \frac{x}{c} + \frac{b}{\sqrt{(cc+aa)}}$, hincque

$$y = \int p\,dx = \frac{xx}{2c} + \frac{bx}{\sqrt{(cc+aa)}} + f.$$

Posteriori casu quo $x = \frac{-b}{(1+aaqq)^{\frac{3}{2}}}$ fit

$$p = \frac{-bq}{(1+aaqq)^{\frac{3}{2}}} + \frac{bq}{\sqrt{(1+aaqq)}} = \frac{aabq^{3}}{(1+aaqq)^{\frac{3}{2}}}.$$

At est

$$dx = \frac{+3aabq\,dq}{(1+aaqq)^{\frac{5}{2}}},\ \text{hincque}$$

$$dy = p\,dx = \frac{3a^{4}bbq^{4}\,dq}{(1+aaqq)^{4}},$$

et ope reductionum

$$y = \frac{-\frac{1}{2}bbq - aabbq^{3}}{(1+aaqq)^{3}} + \frac{1}{2}bb\int\frac{dq}{(1+aaqq)^{3}}.$$

Est vero

$$\int\frac{dq}{(1+aaqq)^{n+1}} = \frac{q}{2n(1+aaqq)^{n}} + \frac{2n-1}{2n}\int\frac{dq}{(1+aaqq)^{n}}.$$

Ergo

Ergo

$$\int\frac{dq}{(1+aaqq)^3}=\frac{q}{4(1+aaqq)^2}+\frac{3}{4}\int\frac{dq}{(1+aaqq)^2}\quad\text{et}$$

$$\int\frac{dq}{(1+aaqq)^2}=\frac{q}{2(1+aaqq)}+\frac{1}{2}\int\frac{dq}{1+aaqq}=\frac{q}{2(1+aaqq)}+\frac{1}{2a}\text{Ang.tang.}aq.$$

Hinc

$$\int\frac{dq}{(1+aaqq)^3}=\frac{q}{4(1+aaqq)^2}+\frac{3q}{8(1+aaqq)}-\frac{3}{8a}\text{Ang. tang.}aq$$

ideoque

$$y=\frac{-bbq(1+2aaqq)}{2(1+aaqq)^2}-\frac{bbq}{8(1+aaqq)^2}+\frac{3bbq}{16(1+aaqq)}+\frac{3bb}{16a}\text{Ang. tang.}aq$$

exiftente

$$x=\frac{-b}{(1+aaqq)^{\frac{1}{2}}},$$

vnde fit

$$1+aaqq=\sqrt{\frac{bb}{xx}}$$

ita vt hoc modo aequatio inter x et y exhiberi poffit. Hoc autem integrale, vt fupra vidimus, tantum eft particulare.

Problema 96.

769. Pofito $dy=pdx$ et $dp=qdx$, fi detur aequatio quaecunque inter y, p et q, ita vt variabilis x ipfa in ea defit, inueftigare aequationem integralem inter x et y.

Solutio.

Cum fit $q=\frac{dp}{dx}$ et $dx=\frac{dy}{p}$ erit $q=\frac{pdp}{dy}$, in aequatione ergo inter y, p et q vbique loco q fubftituatur ifte valor $\frac{pdp}{dy}$, atque habebitur aequatio diffe-

differentialis primi gradus binas tantum variabiles p et y inuoluens, cuius resolutionem per methodos supra expositas tentari oportet. Inuenta autem aequatione integrali inter p et y, inde vel p per y, vel y per p definiatur, quo facilius altera integratio institui possit. Si y per p commode definiri queat, vt y aequetur functioni cuipiam ipsius p, quae sit $=P$ vt sit $y=P$, erit $dx=\frac{dP}{p}$, hincque $x=\int\frac{dP}{p}=\frac{P}{p}+\int\frac{Pdp}{pp}$. Sin autem commodius p per y definire liceat, vt sit $p=Y$ denotante Y functionem quampiam ipsius y, ob $dx=\frac{dy}{p}$, habebitur $x=\int\frac{dy}{Y}$. At si neutrum succedat, nouam variabilem u introducendo per eam vtraque quantitas p et y definiatur, vt fiat $p=U$ et $y=V$, existentibus U et V functionibus ipsius u, atque hinc erit $dx=\frac{dV}{U}$, et $x=\int\frac{dV}{U}$; hocque modo per duplicem integrationem integrale completum obtinebitur.

Coroll. 1.

770. Huiusmodi ergo aequationum differentio-differentialium resolutio quoque reuocatur ad aequationem differentialem primi gradus, cuius resolutio si fuerit in potestate, simul illius integrale exhiberi poterit.

Coroll. 2.

771. Si aequatio inter y, p et q ita fuerit comparata, vt ex ea commode valor ipsius q elici queat, hincque q aequetur functioni ipsarum y et p quae

quae sit T, erit $pdp = Tdy$, quae est aequatio differentialis primi gradus simplex.

Coroll. 3.

772. Sin autem huiusmodi euolutio non succedat, dum littera q vel ad altiores potestates exsurgit, vel signis radicalibus inuoluitur, vel adeo transcendenter ingreditur, aequatio differentialis quidem erit primi gradus sed complicata, quae methodis supra expositis erit tractanda.

Scholion 1.

773. Cum paucis casibus aequationes differentiales primi gradus integrari queant, eosdem etiam hic notasse et per exempla illustrasse iuuabit. Interim vero et reliquos casus quasi solutos spectari conuenit, quandoquidem in aequationibus differentialibus altiorum ordinum id potissimum desideratur vt earum resolutio ad ordinem inferiorem reducatur. Perpetuo enim in Analysi quae ordine tractationis praecedunt, tanquam penitus confecta spectari solent, etiamsi plurima adhuc desiderentur, vt hoc modo multitudo desideratorum diminuatur. Ita quamuis longe adhuc absit, quominus aequationes algebraicas omnium ordinum resoluere valeamus, dum adeo vires nostrae non vltra quartum extenduntur, tamen in Analysi sublimiori omnium istarum aequationum resolutionem pro cognita habemus. Quod etiam vsu non caret, cum in praxi resolutio per approximationem,

quam quousque lubuerit, extendere licet, sufficere possit. Simili modo etiam, quoniam methodum tradidimus aequationum differentialium primi gradus integralia proxime inueniendi, merito totum negotium, vt plane confectum, est censendum, si eo resolutionem aequationum differentialium altiorum graduum reducere potuerimus. Quare in hac secunda parte statim atque aequationem differentialem secundi gradus ad primum gradum perduxerimus, totum negotium pro confecto erit habendum.

Scholion 2.

774. Aequationes ergo differentio-differentiales, quae hoc modo ad differentiales primi gradus reducuntur, ita sunt comparatae, vt posito $dy = pdx$ et $dp = qdx$, variabilis x ipsa inde tollatur, et aequatio inter solas tres variabiles y, p et q oriatur. Casus ergo quibus talis aequatio resolutionem admittit, duplicis sunt generis, ad quorum prius referendi sunt ii, quibus q vnicam obtinet dimensionem, vnde q functioni cuipiam ipsarum y et p aequari potest. Cum igitur sit $q = \frac{pdp}{dy} = f:(y \text{ et } p)$, quam ponamus $= T$, resolutio succedet: 1) Si T sit functio homogenea vnius dimensionis ipsarum y et p. 2) Si fuerit $T = \frac{P}{y+Q}$, designantibus P et Q functiones quascunque ipsius p tantum, hinc enim fit $Pdy = ypdp + Qpdp$; quorsum etiam refertur casus, $T = \frac{P}{y + Q y^n}$. 3) Si fuerit $T = p(Yp + Z)$, si

 qui-

quidem Y et Z sint functiones quaecunque ipsius y, quia tum aequatio $dp = Ypdy + Zdy$, ob vnicam dimensionem ipsius p est integrabilis, quorsum etiam referendus est casus $T = p(Yp + Zp^2)$. Pro altero genere si quantitas q plures habeat dimensiones, vel signis radicalibus sit implicata, vel adeo transcendenter ingrediatur, aequatio inter y, p et q, resolutionem admittet: 1) Si posito $q = pu$, vt sit $u = \frac{dp}{dy}$, aequatio resultet homogenea inter y et p, inqua scilicet y et p vbique eundem dimensionum numerum compleant, vtcunque caeterum u in eam ingrediatur. 2) Si in aequatione post substitutionem $q = pu$, inter y, p et u orta, altera quantitas y vel p vnicam obtineat dimensionem. 3) Si posito $y = v^{\mu}$, $p = z^{\mu+\nu}$ et $u = t^{\nu}$ aequatio oriatur homogenea inter ternas quantitas v, z et t, huiusmodi enim aequationes supra resoluere docuimus.

Exemplum I.

775. *Posito elemento* dx *constante, si habeatur haec aequatio differentio-differentialis:*

$$ddy + Adxdy + Bydx^2 = 0$$

eius integrale completum inuenire.

Posito $dy = pdx$ et $dp = qdx$, aequatio nostra erit

$$q + Ap + By = 0, \text{ seu } pdp + Apdy + Bydy = 0$$

quae

quae cum fit homogenea pofito $p = vy$ abit in

$$vvydy + vyydv + Avydy + Bydy = 0$$

vnde fit

$$\frac{dy}{y} + \frac{vdv}{vv + Av + B} = 0.$$

Sit $vv + Av + B = (v+\alpha)(v+\beta)$
vt fit $\alpha + \beta = A$ et $\alpha\beta = B$ erit

$$\frac{dy}{y} + \frac{\alpha dv}{(\alpha-\beta)(v+\alpha)} - \frac{\beta dv}{(\alpha-\beta)(v+\beta)} = 0$$

hincque integrando:

$$ly + \frac{\alpha}{\alpha-\beta} l(v+\alpha) - \frac{\beta}{\alpha-\beta} l(v+\beta) = C \text{ feu}$$

$$y = a(v+\beta)^{\frac{\beta}{\alpha-\beta}}(v+\alpha)^{\frac{-\alpha}{\alpha-\beta}}, \text{ ideoque}$$

$$p = vy = av(v+\beta)^{\frac{\beta}{\alpha-\beta}}(v+\alpha)^{\frac{-\alpha}{\alpha+\beta}}.$$

Tum vero eft

$$dx = \frac{dy}{p} = \frac{dy}{vy}, \text{ vnde ob}$$

$$\frac{dy}{y} = \frac{-vdv}{vv + Av + B}, \text{ erit}$$

$$dx = \frac{-dv}{vv + Av + B} = \frac{dv}{(\alpha-\beta)(v+\alpha)} - \frac{dv}{(\alpha-\beta)(v+\beta)}, \text{ et}$$

$$x = \frac{1}{\alpha-\beta} l \frac{v+\alpha}{v+\beta} + \text{Conft.}$$

Verum haec refolutio fit facilior fequenti modo: Cum fit

$$\frac{dy}{y} = \frac{-vdv}{(v+\alpha)(v+\beta)} \text{ et } dx = \frac{-dv}{(v+\alpha)(v+\beta)} \text{ erit}$$

$$\frac{dy}{y} + \alpha dx = \frac{-dv}{v+\beta} \text{ et } \frac{dy}{y} + \beta dx = \frac{-dv}{v+\alpha}, \text{ erit}$$

$$ly + \alpha x = la - l(v+\beta) \text{ et } ly + \beta x = lb - l(v+\alpha).$$

Ergo $v+\beta=\frac{a}{y}e^{-\alpha x}$ et $v+\alpha=\frac{b}{y}e^{-\beta x}$ vnde fit

$$\alpha-b=\tfrac{1}{y}(\beta e^{-\beta x}-a e^{-\alpha x})$$

ideoque mutatis constantibus:

$$y=\mathfrak{A}e^{-\alpha x}+\mathfrak{B}e^{-\beta x},$$

quae integratio locum habet, si α et β sint quantitates reales et inaequales. Cum igitur posuerimus

$$vv+Av+B=(v+\alpha)(v+\beta) \text{ erit}$$

$$\alpha=\tfrac{1}{2}A+\sqrt{(\tfrac{1}{4}AA-B)} \text{ et } \beta=\tfrac{1}{2}A-\sqrt{(\tfrac{1}{4}AA-B)}$$

hinc prout expressio $\frac{1}{4}AA-B$ fuerit vel positiua, vel negatiua, vel euanescens, tres habebimus casus euoluendos:

1) Sit $\frac{1}{2}A=m$ et $\sqrt{(\frac{1}{4}AA-B)}=n$, erit aequationis propositae integrale completum:

$$y=\mathfrak{A}e^{-(m+n)x}+\mathfrak{B}e^{-(m-n)x}=e^{-mx}(\mathfrak{A}e^{-nx}+\mathfrak{B}e^{nx}).$$

2) Sit $\frac{1}{2}A=m$ et $\sqrt{(\frac{1}{4}AA-B)}=n\sqrt{-1}$, ob $e^{nx\sqrt{-1}}=\text{cos}.nx+\sqrt{-1}.\text{sin}\,nx$ et $e^{-nx\sqrt{-1}}=\text{cos}.nx-\sqrt{-1}.\text{sin}.nx$ erit constantibus mutandis:

$$y=e^{-mx}(\mathfrak{C}\,\text{cos}.nx+\mathfrak{D}\,\text{sin}.nx)=\mathfrak{C}e^{-mx}\text{cos}.(nx+\epsilon)$$

3) Sit $\frac{1}{2}A=m$ et $\sqrt{(\frac{1}{4}AA-B)}=0$, seu in casu primo $n=0$, ob $e^{-nx}=1-nx$ et $e^{nx}=1+nx$ fiet

$$y=e^{-mx}(\mathfrak{C}+\mathfrak{D}x).$$

Coroll. 1.

Coroll. 1.

776. Ad aequationis ergo propositae integrale inueniendum, aequationis $vv + Av + B = 0$ radices inuestigari oportet, quibus inuentis facile erit integrale completum assignare.

Coroll. 2.

777. Haec autem aequatio quadratica $vv + Av + B = 0$ insignem habet analogiam cum ipsa aequatione proposita $ddy + Adydx + Bydx^2 = 0$, ex qua quippe oritur scribendo 1, v, v^2 loco y, $\frac{dy}{dx}$ et $\frac{ddy}{dx^2}$.

Coroll. 3.

778. Formata autem aequatione hac algebraica $vv + Av + B = 0$, si eius factor sit $v + \alpha$, ex eo statim integrale particulare deducitur $y = \mathfrak{A}e^{-\alpha x}$, similiterque alter factor $v + \beta$ integrale particulare dabit $y = \mathfrak{B}e^{-\beta x}$, quibus coniunctis obtinetur integrale completum $y = \mathfrak{A}e^{-\alpha x} + \mathfrak{B}e^{-\beta x}$.

Scholion.

779. Infra methodus facilior tradetur huiusmodi aequationes differentio-differentiales tractandi, quae adeo ad talem formam

$$ddy + Pdydx + Qydx^2 = 0$$

patet, vbi P et Q sint functiones quaecunque

quae cum sit homogenea ponatur $p = \frac{y}{u}$ erit

$$y\,dy\sqrt{(1+\frac{aa}{uu})} = \frac{aby}{u^3}(u\,dy - y\,du) \text{ seu}$$

$$uu\,dy\sqrt{(aa+uu)} = abu\,dy - aby\,du,$$

vnde fit

$$\frac{dy}{y} = \frac{ab\,du}{abu - uu\sqrt{(aa+uu)}}.$$

Ponatur $\sqrt{(aa+uu)} = su$, erit $uu = \frac{aa}{ss-1}$,

$$\frac{du}{u} = \frac{-s\,ds}{ss-1} \text{ et } \frac{dy}{y} = \frac{-bs\,ds}{bss-as-b} = \frac{-s\,ds}{ss-2ns-1}$$

posito $\frac{a}{b} = 2n$. Ergo

$$\frac{2dy\sqrt{(nn+1)}}{y} = -\frac{ds(n+\sqrt{(nn+1)})}{s-n-\sqrt{(nn+1)}} + \frac{ds(n-\sqrt{(nn+1)})}{s-n+\sqrt{(nn+1)}}$$

ideoque

$$y^{2\sqrt{(nn+1)}} = \frac{C(s-n+\sqrt{(nn+1)})^{n-\sqrt{(nn+1)}}}{(s-n-\sqrt{(nn+1)})^{n+\sqrt{(nn+1)}}}.$$

Datur igitur y per s, vt sit $y = S$, hincque

$$u = \frac{a}{\sqrt{(ss-1)}} \text{ et } p = \frac{S\sqrt{(ss-1)}}{a}, \text{ atque}$$

$$dx = \frac{a\,dS}{S\sqrt{(ss-1)}} \text{ seu } dx = \frac{-as\,ds}{(ss-2ns-1)\sqrt{(ss-1)}}$$

quae formula ad rationalitatem perduci et per logarithmos seu arcus circulares integrari potest.

Exemplum 3.

781. *Posito* $dy = p\,dx$ *et* $dp = q\,dx$, *inuenire integrale huius aequationis* $\frac{(pp+yy)\sqrt{(pp+yy)}}{2pp+yy-qy} = ny$.

 Cum

Cum fit $q = \frac{pdp}{dy}$ erit

$$dy(pp+yy)\sqrt{(pp+yy)} = 2npp y dy + ny^3 dy - nyypdp$$

ob cuius homogeneitatem ponatur $p = uy$, fietque

$$y^3 dy(uu+1)^{\frac{3}{2}} = 2nuuy^3 dy + ny^3 dy - nu^2 y^3 dy - nuy^4 du$$

vnde colligitur:

$$\frac{dy}{y} = \frac{-nudu}{(uu+1)\sqrt{(1+uu)} - nuu - n} = \frac{nudu}{(uu+1)(n - \sqrt{(1+uu)})}$$

et y per u definitur; ex quo erit $p = uy$ et

$$dx = \frac{dy}{uy} = \frac{ndu}{(uu+1)(n - \sqrt{(1+uu)})}$$

Cafu quo $n = 1$ erit:

$$\frac{dy}{y} = \frac{-udu}{(uu+1)(\sqrt{(uu+1)} - 1)} = \frac{du(1+\sqrt{(1+uu)})}{u(uu+1)}$$

et $dx = \frac{du(1+\sqrt{(1+uu)})}{uu(uu+1)}$. Eft vero

$$\int \frac{du}{u(uu+1)} = l \frac{u}{\sqrt{(uu+1)}};\ \int \frac{du}{uu(uu+1)} = -\frac{1}{u} - \text{Ang. tang. } u$$

$$\int \frac{du}{u\sqrt{(uu+1)}} = l \frac{\sqrt{(uu+1)} - 1}{u};\ \int \frac{du}{uu\sqrt{(uu+1)}} = -\frac{\sqrt{(uu+1)}}{u}$$

vnde colligitur

$$y = C\frac{\sqrt{(uu+1)} - 1}{\sqrt{(uu+1)}} = C\left(1 - \frac{1}{\sqrt{(uu+1)}}\right) \text{ et}$$

$$x = D - \frac{1 - \sqrt{(uu+1)}}{u} - \text{Ang. tang. } u.$$

Inde eft

$$\frac{1}{\sqrt{(uu+1)}} = 1 - \frac{y}{a};\ \text{et } u = \frac{\sqrt{(2ay - yy)}}{a - y}$$

ideoque

$$x = D - \sqrt{\frac{2a - y}{y}} - \text{Ang. cof. } \frac{a - y}{a}$$

quae

quae formulae introducendo angulum Φ cuius cofinus eft $\frac{a-y}{a}$ ita commodius exhibentur

$$y = a(1 - \operatorname{cof.}\Phi) \text{ et } x = \zeta - \Phi - \operatorname{cot.}\tfrac{1}{2}\Phi.$$

Coroll. 1.

782. Ex aequatione feparata primum inuenta folutio particularis eruitur tribuendo ipfi u eiusmodi valorem conftantem, vt denominator euanefcat, qui eft $u = \sqrt{(nn-1)}$ hinc $p = y\sqrt{(nn-1)}$ et $dx\sqrt{(nn-1)} = \frac{dy}{y}$, vnde fit $ly = la + x\sqrt{(nn-1)}$.

Coroll. 2.

783. Cafu quo $n = 1$, hic cafus particularis praebet $y = a$, pro valore quocunque alterius variabilis; fit enim $u = 0$, ideoque et $p = 0$, ita vt ex aequatione $dy = pdx$ quantitas x non determinetur.

Scholion.

784. Si y defignet radium vectorem ex puncto fixo ad curuam quampiam ductum, et x angulum, quem ifte radius cum recta quadam pofitione data conftituit, formula $\frac{(pp+yy)\sqrt{(pp+yy)}}{2pp+yy-qy}$ exprimit huius curuae radium curucdinis. In exemplo ergo propofito eiusmodi quaeritur curua, cuius radius curuedinis aequetur ipfi ny, cui quaeftioni cafu $n = 1$ vtique fatisfacit valor $y = a$ qui praebet circulum

culum, qui etiam ex aequatione integrali colligitur $y = C\left(1 - \frac{1}{\sqrt{(uu+1)}}\right)$, sumendo constantem C infinitam, tum enim necesse est sit $u = 0$ et $p = 0$ sicque angulus x non determinatur. Praeter circulum autem infinitae aliae lineae curuae satisfaciunt. At si $n > 1$, solutio particularis $ly = la + x\sqrt{(nn-1)}$ praebet spiralem logarithmicam, praeter quam autem etiam infinitae aliae curuae satisfaciunt; casibus autem $n < 1$ nulla huiusmodi solutio particularis locum habet, sed formulas pro $\frac{dy}{y}$ et dx inuentas reuera integrari oportet.

Exemplum 4.

785. *Posito* $dy = pdx$ *et* $dp = qdx$ *inuenire relationem inter* x *et* y, *vt fiat* $\frac{(pp+yy)\sqrt{(pp+yy)}}{2pp+yy-qy} = a$.

Cum sit $q = \frac{pdp}{dy}$, ponatur $pp + yy = zz$, ob $pdp = qdy$ erit $qdy + ydy = zdz$ seu $q + y = \frac{zdz}{dy}$. Aequatio autem proposita induit hanc formam:

$$z^3 = a(2zz - yy - qy) = a\left(2zz - \frac{yzdz}{dy}\right) \text{ seu}$$

$$zzdy = 2azdy - aydz$$

vnde fit

$$\frac{dy}{y} = \frac{adz}{2az - zz} \text{ seu } \frac{2dy}{y} = \frac{dz}{z} + \frac{dz}{2a - z}$$

quare integrando colligitur:

$$yy = \frac{Cz}{2a - z} \text{ et } pp = zz - \frac{Cz}{2a - z} = \frac{-Cz + 2azz - z^3}{2a - z}.$$

At

At eſt $z = \frac{2ayy}{C+yy}$, ergo

$$pp = \frac{4aay^4}{(C+yy)^2} - yy = \frac{yy(4aayy - (C+yy)^2)}{(C+yy)^2}.$$

Hinc igitur oritur

$$dx = \frac{(C+yy)\,dy}{y\sqrt{(4aayy-(C+yy)^2)}}$$

fit $yy = u$ erit

$$dx = \frac{(C+u)\,du}{2u\sqrt{(4aau-(C+u)^2)}}.$$

Haec aequatio tractabilior redditur ponendo

$$u = 2aa - C + 2a\,\mathrm{cof.}\,\Phi\sqrt{(aa-C)}$$

fit enim

$$dx = \frac{-ad\Phi(a+\mathrm{cof.}\,\Phi\sqrt{(aa-C)})}{2aa-C+2a\,\mathrm{cof.}\,\Phi\sqrt{(aa-C)}} \text{ feu}$$

$$2dx = -d\Phi - \frac{C\,d\Phi}{2aa-C+2a\,\mathrm{cof.}\,\Phi\sqrt{(aa-C)}}$$

quae integrata dat

$$2x = \zeta - \Phi - \text{Ang. cof.}\,\frac{m+\mathrm{cof.}\,\Phi}{1+m\,\mathrm{cof.}\,\Phi}$$

poſito $m = \frac{2a\sqrt{(aa-C)}}{2aa-C}$, vt fit

$$C = \frac{2aa\sqrt{(1-mm)}}{1+\sqrt{(1-mm)}} \text{ et } \sqrt{(aa-C)} = \frac{ma}{1+\sqrt{(1-mm)}}$$

hincque $yy = \frac{2aa(1+m\,\mathrm{cof.}\,\Phi)}{1+\sqrt{(1-mm)}}$

vnde fit

$$\mathrm{cof.}\,\Phi = \frac{yy(1+\sqrt{(1-mm)})-2aa}{2maa} \text{ et } \frac{m+\mathrm{cof.}\,\Phi}{1+m\,\mathrm{cof.}\,\Phi} = \frac{yy(1+\sqrt{(1-mm)})-2aa(1-mm)}{myy(1+\sqrt{(1-mm)})}$$

Coroll. I.

786 Cum ſit $yy = \frac{2aa(1+m\,\mathrm{cof.}\,\Phi)}{1+\sqrt{(1-mm)}}$ erit $yy = aa + bb + 2ab\,\mathrm{cof.}\,\Phi$ ſi ponatur $b = \frac{a(1-\sqrt{(1-mm)})}{m}$, vnde fit

$$m = \frac{2ab}{aa+bb} \text{ et } \sqrt{(1-mm)} = \frac{aa-bb}{aa+bb}$$

hincque

$$2x = \zeta - \Phi - \text{Ang. cof.}\, \frac{2ab+(aa+bb)\,\mathrm{cof.}\,\Phi}{aa+bb+ab\,\mathrm{cof.}\,\Phi} \text{ ſeu}$$

$$2x = \zeta - \Phi - \text{Ang. ſin.}\, \frac{(aa-bb)\,\mathrm{ſin.}\,\Phi}{yy}.$$

Coroll. 2.

787. Si vt ſupra radius vector y cum angulo x referatur ad lineam curuam, hanc curuam circulum eſſe oportet radio $= a$ deſcriptum. Fit autem $dx = \frac{d\Phi(aa - ab\,\mathrm{cof.}\,\Phi)}{aa+bb-2ab\,\mathrm{cof.}\,\Phi}$ ſumto $yy = aa + bb - 2ab\,\mathrm{cof.}\,\Phi$, hincque $x = \zeta + \text{Ang. tang.}\, \frac{a\,\mathrm{ſin.}\,\Phi}{a\,\mathrm{cof.}\,\Phi - b}$ cuius applicatio ad Geometriam rem facit perſpicuam.

Exemplum 5.

758. *Sumto elemento* dx *conſtante, ſi proponatur haec aequatio* $ddy(ydy + adx) = dy(dx^2 + dy^2)$, *eius integrale inuenire.*

Poſito $dy = p\,dx$ et $dp = q\,dx$ habebimus:

$q(py + a) = p(1 + pp)$ et ob $q = \frac{p\,dp}{dy}$

$dp(py + a) = dy(1 + pp)$ ſiue

$dy - \frac{py\,dp}{1+pp} = \frac{a\,dp}{1+pp}$ quae

quae integrata dat

$$\frac{y}{\sqrt{(1+pp)}} = \frac{ap}{\sqrt{(1+pp)}} + b \text{ ideoque}$$

$$y = ap + b\sqrt{(1+pp)} \text{ et } x = \int\frac{dy}{p} = alp + bl(p + \sqrt{(1+pp)}) + c$$

ita vt x et y per eandem variabilem p exprimantur. Si constans b sumatur $=0$, obtinetur integrale particulare

$$y = ap \text{ et } x = alp + C = al\tfrac{y}{a} + C$$

seu in exponentialibus $y = Ce^{x:a}$. Sin autem sumatur $b = a$ ob

$$p + \sqrt{(1+pp)} = \tfrac{y}{a} \text{ et } p = \tfrac{yy-aa}{2ay} \text{ erit}$$

$$x = al\tfrac{yy-aa}{2aa} + C \text{ seu } yy = aa + Ce^{x:a}.$$

Exemplum 6.

789. *Sumto* dx *constante huius aequationis differentio-differentialis* $dy^2 - yddy = n\sqrt{(dx^2dy^2 + aaddy^2)}$ *integrale inuenire.*

Posito $dy = pdx$ et $dp = qdx$ erit

$$pp - qy = n\sqrt{(pp + aaqq)}$$

quae facto $q = pu$, vt sit $\frac{pdp}{dy} = pu$, ideoque $dp = udy$ abit in

$$pp - puy = np\sqrt{(1 + aauu)} \text{ seu}$$

$$p - uy = n\sqrt{(1 + aauu)}.$$

Iam quia $dp = udy$ differentietur haec aequatio, prodibitque

$$-ydu = \frac{naaudu}{\sqrt{(1+aauu)}} \qquad \text{hinc}$$

hinc vel $du = 0$ vel $y = \frac{-naau}{\sqrt{(1+aauu)}}$.

1) Casu $du = 0$ fit $u = \alpha$, $p = \alpha y + \beta$, et $dx = \frac{dy}{\alpha y + \beta}$, hincque $\alpha x = l(\alpha y + \beta) + C$.

2) Si $y = \frac{-naau}{\sqrt{(1+aauu)}}$ erit

$p = uy + n\sqrt{(1+aauu)} = \frac{n}{\sqrt{(1+aauu)}}$ hincque

$dx = \frac{dy}{p} = \frac{-aadu}{1+aauu}$ et $x = -a \operatorname{Ang. tang.} au + C$

vel ob $u = \frac{y}{a\sqrt{(nnaa-yy)}}$, aequatio inter x et y quaesita erit

$\frac{b-x}{a} = \operatorname{Ang. tang.} \frac{y}{\sqrt{(nnaa-yy)}} = \operatorname{Ang. sin.} \frac{y}{na}$

vnde fit $y = na \sin. \frac{b-x}{a}$. Haec autem relatio tantum pro integrali particulari est habenda.

CAPVT III.

CAPVT III.

DE AEQVATIONIBVS DIFFERENTIO - DIFFERENTIALIBVS HOMOGENEIS ET QVAE AD EAM FORMAM REDVCI POSSVNT.

Problema 97.

790.

Aequationum differentio - differentialium homogenearum naturam explicare, atque ad formam finitam ponendo $dy = pdx$ et $dp = qdx$ accommodare.

Solutio.

Sumto elemento dx constante aequatio differentio - differentialis vulgari modo expressa dicitur homogenea, si non solum ipsis variabilibus x et y, sed etiam earum differentialibus dx et dy, itemque ipsi ddy, singulis vnam dimensionem tribuendo, omnes aequationis termini eundem dimensionum numerum contineant; veluti in hac aequatione:

$$xxddy + xdx^2 + ydy^2 = 0,$$

vbi in singulis terminis ternae insunt dimensiones.

Quodſi ergo ponamus $\frac{dy}{dx}=p$, ac $\frac{dp}{dx}=\frac{ddy}{dx^2}=q$, littera p nullam dimenſionem, littera vero q vnam dimenſionem negatiuam continere erit cenſenda. Hinc aequatio differentio - differentialis ad formam hic receptam reducta, vt nonniſi quantitates finitas x, y, p et q contineat, erit homogenea, ſi litteris x et y vnam dimenſionem tribuendo, litterae p vero nullam, at litterae q vnam dimenſionem negatiuam, in ſingulis aequationis terminis idem oriatur dimenſionum numerus. Viciſſim ergo quoties haec proprietas in aequatione inter quaternas quantitates x, y, p et q propoſita deprehenditur, ea aequatio erit homogenea et forma vulgari expreſſa manifeſto homogeneitatem prae ſe feret.

Coroll. 1.

791. Si ergo in aequatione tali homogenea inter x, y, p et q ſtatuatur $y=ux$ et $q=\frac{v}{x}$, omnes termini eandem poteſtatem ipſius x continebunt, qua ergo per diuiſionem ſublata, aequatio prodibit tres tantum variabiles u, v et p inuoluens.

Coroll. 2.

792. Criterium igitur aequationis homogeneae inter quatuor quantitates x, y, p et q propoſitae in hoc conſiſtit, vt poſito $y=ux$ et $q=\frac{v}{x}$, quantitas x prorſus ex calculo exterminetur.

Coroll. 3.

Coroll. 3.

793. Facta itaque hac fubftitutione, qua obtinetur aequatio inter ternas quantitates u, v et p, ex ea pro lubitu vel p per u et v, vel v per u et p, vel u per v et p definiri poterit.

Scholion.

794. Simili modo ideam homogeneitatis in aequationibus differentio-differentialibus conftituimus, quo in aequationibus differentialibus primi gradus fumus vfi. In his quidem, cum differentialia fponte eundem dimenfionum numerum conftituere debeant, homogeneitas ex folis ipfis variabilibus x et y diiudicatur. At in aequationibus differentio-differentialibus praeter ipfas variabiles x et y etiam litterae q ratio in computo dimenfionum haberi debet, ita tamen vt ipfi vna dimenfio negatiua fit tribuenda; littera autem p in hunc computum plane non ingreditur, quae ergo vtcunque aequationi implicetur, homogeneitatem non turbat. Plurimum autem intereft probe noffe indolem differentio-differentialium aequationum homogenarum, cum earum refolutio ad refolutionem aequationum differentialium primi gradus reduci poffit, ita vt fi haec fucceflerit, etiam ipfarum aequationum differentio-differentialium integratio habeatur, id quod in fequenti problemate luculentius oftendemus.

Problema 98.

795. Propofita aequatione differentio-differentiali homogenea, eius refolutionem ad integrationem aequationis differentialis primi gradus reducere.

Solutio.

Reducta aequatione ponendo $dy = pdx$ et $dp = qdx$ ad formam hic receptam vt habeatur aequatio inter quatuor quantitates finitas x, y, p et q, ponatur $y = ux$ et $q = \frac{v}{x}$, ac cum aequatio fit homogenea, hoc modo quantitas x penitus ex calculo elidetur, ita vt proditura fit aequatio inter ternas quantitates u, v et p, ex qua vnam per binas reliquas definire liceat. Nunc igitur cum fit $dy = pdx$ erit $udx + xdu = pdx$ hincque $\frac{dx}{x} = \frac{du}{p-u}$. Deinde ob $dp = qdx$ erit $dp = \frac{vdx}{x}$, ideoque $\frac{dx}{x} = \frac{dp}{v}$; ex quo dupl ci ipfius $\frac{dx}{x}$ valore colligitur $\frac{du}{p-u} = \frac{dp}{v}$ feu $vdu = pdp - udp$. Quodfi ergo ex illa aequatione quantitas v definiatur per binas p et u, habebitur aequatio differentialis primi gradus inter binas variabiles p et u, cuius integratio fi fuerit in poteftate, vt p per u innotefcat, aequatio $\frac{dx}{x} = \frac{du}{p-u}$, in qua variabiles x et u funt feparatae, integretur, ficque x per u definietur, vnde fit $y = ux$; feu ftatim in hoc integrali loco u fcribatur $\frac{y}{x}$, et habebitur aequatio inter x et y quaefita.

Coroll. 1.

Coroll. 1.

796. Totum ergo negotium reducitur ad integrationem huius aequationis differentialis simplicis $vdu = pdp - udp$, quae si ope regularum supra traditarum expediri queat, simul aequationis differentio-differentialis integratio habetur.

Coroll. 2.

797. Simul autem patet resolutionem huiusmodi aequationum duplicem integrationem requirere, vnde duae quantitates arbitrariae constantes ingrediuntur, quibus integrale completum constituitur.

Coroll. 3.

798. Etiamsi autem integratio $vdu = pdp - udp$ non succedat, tamen ingens lucrum est rem eo perduxisse, cum supra methodus generalis sit tradita integralia omnium aequationum differentialium primi gradus proxime assignandi.

Scholion.

799. Operae igitur pretium erit eos casus perpendere, quibus aequatio $vdu = pdp - udp$ integrationem admittit; quamobrem examinemus, qualis functio v debeat esse ipsarum p et u, vt hoc eueniat. Primum autem patet hoc fieri, si v fuerit functio homogenea vnius dimensionis ipsarum p et u, quoniam tum ipsa haec aequatio fit homogenea ac

per regulas supra expositas ad integrationem perduci potest. Deinde etiam integratio succedit si fuerit v functio quaecunque ipsius p, quoniam tum altera variabilis u vnam dimensionem non superat, et aequationis $du+\frac{udp}{v}=\frac{pdp}{v}$ integrale est $e^{\int\frac{dp}{v}}u=\int e^{\int\frac{dp}{v}}\frac{pdp}{v}$. Tertio integrationem absoluere licebit, si v fuerit functio quaecunque quantitatis $p-u$. Posito enim $p-u=s$, vt sit v functio ipsius s, ob $p=s+u$ nostra aequatio erit $vdu=sds+sdu$, ideoque $du=\frac{sds}{v-s}$ et $u=\int\frac{sds}{v-s}$, quae integratio adeo ad formulas simplices est referenda. Quarto manente $s=p-u$, si P, Q, R denotent functiones quascunque ipsius s aequatio nostra $vdu=sds+sdu$ tractari poterit, si fuerit $v=s+\frac{Ps}{Qu+Ru^n}$, tum enim fit $Pdu=Quds+Ru^n ds$. Quinto etiam patet, si denotantibus V et U functiones quascunque ipsius u, fuerit $v=s+Vss+Us^n$, integrationem quoque fore in potestate, fit enim aequatio nostra $Vsdu+Us^{n-1}du=ds$. Atque in genere si aequatio differentialis $ds=Zdu$ fuerit integrabilis, existente Z functione binarum variabilium s et u, cum nostra aequatio fit $sds=(v-s)du$, habebimus $v=s+Zs$ pro omnibus casibus integrationem admittentibus.

Exemplum 1.

800. *Sumto elemento* dx *constante, si proponatur haec aequatio* $xxddy=xdxdy+nydx^2$ *eius integrale inuenire.*

Posito

Pofito $dy=pdx$ et $dp=qdx$, erit $qxx=px+ny$, vnde facto $y=ux$ prodit $qx=p+nu=v$, ita vt v fit functio vnius dimenfionis ipfarum p et u, et aequatio noftra $(p+nu)du=pdp-udp$ fiat homogenea. Cum ergo fit $nudu+pdu+udp=pdp$, erit integrando

$$C+nuu+2pu=pp \quad \text{et} \quad p=u+\sqrt{(C+(n+1)uu)}.$$

Habebimus ergo $\frac{dx}{x}=\frac{du}{\sqrt{(C+(n+1)uu)}}$, quae denuo integrata dat

$$lx=\frac{1}{\sqrt{(n+1)}}\, l\frac{u\sqrt{(n+1)}+\sqrt{(C+(n+1)uu)}}{D} \quad \text{feu}$$

$$Dx^{\sqrt{(n+1)}}=u\sqrt{(n+1)}+\sqrt{(C+(n+1)uu)}$$

hincque

$$D^2x^{2\sqrt{(n+1)}}-2Dx^{\sqrt{(n+1)}}u\sqrt{(n+1)}=C$$

fit $D=f\sqrt{(n+1)}$ et $C=g(n+1)$ vt habeatur

$$ffx^{2\sqrt{(n+1)}}-2fx^{\sqrt{(n+1)}}u=g, \quad \text{exiftente } u=\frac{y}{x}.$$

Cafu quo $n=-1$, ob $\frac{dx}{x}=\frac{du}{\alpha}$, erit $\alpha lx=u=\frac{y}{x}$ ideoque $y=\alpha x\, l\frac{x}{a}$. At fi $n+1$ fit numerus negatiuus integratio etiam angulos implicabit.

Coroll. 1.

801. Si fit $n=0$, huius aequationis $xxddy=xdxdy$ integrale completum erit $ffx^2-2fy=g$, qui cafus per fe eft perfpicuus. Cum ex $\frac{ddy}{dy}=\frac{dx}{x}$ fluat $\frac{dy}{dx}=fx$ et $2y=fxx-\frac{g}{f}$.

Coroll. 2.

Coroll. 2.

802. Si sit $n=3$, aequationis $xxddy=xdxdy+3ydx^2$ integrale completum est $ffx^4-2fxy=g$. Idem euenit si loco $\sqrt{(n+1)}$ scribatur -2, fit enim $\frac{ff}{x^4}-\frac{2fy}{x^3}=g$, et $ff-2fxy=gx^4$, vtraque redit ad $y=\frac{\alpha}{x}+\beta x^3$.

Exemplum 2.

803. *Sumto elemento* dx *constante., si proponatur baec aequatio* $\frac{xxddy}{dx}=\sqrt{(mxxdy^2+nyydx^2)}$, *eius integrale completum inuenire.*

Ob $dy=pdx$ et $dp=qdx$ habebimus

$qxx=\sqrt{(mppxx+nyy)}$, quae posito $y=ux$ abit in

$qx=\sqrt{(mpp+nuu)}=v$ ob $q=\frac{v}{x}$.

Quia ergo est $\frac{dx}{x}=\frac{du}{p-u}=\frac{dp}{v}$, erit

$$du\sqrt{(mpp+nuu)}=(p-u)dp$$

quae est aequatio homogenea. Ponatur ergo $p=su$ et prodit:

$du\sqrt{(mss+n)}=(s-1)(sdu+uds)$ hincque

$$\frac{du}{u}=\frac{(s-1)ds}{\sqrt{(mss+n)}-ss+s}$$

ex qua fit

$$\frac{dx}{x}=\frac{du}{(s-1)u}=\frac{ds}{\sqrt{(mss+n)}-ss+s}$$

vnde

vnde tum $u = \frac{y}{x}$ quam x per eandem variabilem s determinatur.

Exemplum 3.

804. *Sumto elemento* dx *constante, si proponatur haec aequatio* $nx^2 ddy = (ydx - xdy)^2$ *eius integrale inuenire.*

Posito $dy = pdx$ et $dp = qdx$ erit $nx^2 q = (y - px)^2 = nxxv$ ob $q = \frac{v}{x}$. Si iam statuatur $y = ux$ fiet

$$nv = (u - p)^2 \text{ et } \frac{dx}{x} = \frac{du}{p - u} = \frac{dp}{v} = \frac{ndp}{(p - u)^2}$$

vnde habetur $ndp = pdu - udu$: quae facto $p - u = s$ abit in $ndu + nds = sdu$ seu $du = \frac{nds}{s - n}$, hinc $u = nl\frac{s - n}{\alpha}$. Tum vero ob $p - u = s$ erit

$$\frac{dx}{x} = \frac{du}{s} = \frac{nds}{\beta(s - n)} \text{ et } lx = l\frac{s - n}{\beta s} \text{ ideoque}$$

$$x = \frac{s - n}{\beta s} \text{ at } y = nxl\frac{s - n}{\alpha}.$$

Cum ergo sit

$$s = \frac{n}{1 - \beta x} \text{ erit } y = nxl\frac{n\beta x}{\alpha(1 - \beta x)}.$$

Corollarium.

805. Aequatio $nx^3 q = (y - px)^2$, facilius resoluitur ponendo $y - px = z$, vnde fit $-xdp = dz$; quare ob $qdx = dp$ erit $nx^3 dp = zzdx = -nxxdz$, ideoque $\frac{1}{x} = \frac{1}{a} - \frac{n}{z}$ seu $\frac{x - a}{ax} = \frac{n}{y - px}$ ergo $y - px = \frac{nax}{x - a}$ $= \frac{ydx - xdy}{dx}$. Quare $\frac{ydx - xdy}{xx} = \frac{nadx}{x(x - a)}$ et $\frac{y}{x} = nl\frac{x}{x - a} + C$ vt ante.

Exemplum 4.

806. *Sumto* dx *constante*, *si proponatur haec aequatio*

$$(dx^2+dy^2)\sqrt{(dx^2+dy^2)}=ndxddy\sqrt{(xx+yy)}$$

integrale inuenire.

Posito $dy=pdx$ et $dp=qdx$, aequatio proposita hanc induit formam:

$$(1+pp)\sqrt{(1+pp)}=nq\sqrt{(xx+yy)}$$

quae facto $y=ux$ et $q=\frac{v}{x}$ transit in hanc

$$(1+pp)\sqrt{(1+pp)}=nv\sqrt{(1+uu)}.$$

Cum igitur sit $\frac{dx}{x}=\frac{du}{p-u}=\frac{dp}{v}$, ob $v=\frac{(1+pp)\sqrt{(1+pp)}}{n\sqrt{(1+uu)}}$ erit $(1+pp)^{\frac{3}{2}}du=u(p-u)dp\sqrt{(1+nu)}$ cuius resolutio non sponte patet. Calculum autem ad angulos reuocando sit $p=\text{tang.}\,\Phi$ et $u=\text{tang.}\,\omega$ erit $dp=\frac{d\Phi}{\text{cos.}\,\Phi^2}$; $du=\frac{d\omega}{\text{cos.}\,\omega^2}$; $\sqrt{(1+pp)}=\frac{1}{\text{cos.}\,\Phi}$; $\sqrt{(1+uu)}=\frac{1}{\text{cos.}\,\omega}$ et $p-u=\frac{\text{sin.}\,(\Phi-\omega)}{\text{cos.}\,\Phi\,\text{cos.}\,\omega}$, hincque

$$\frac{1}{\text{cos.}\,\Phi^3}\cdot\frac{d\omega}{\text{cos.}\,\omega^2}=\frac{n\,\text{sin.}\,(\Phi-\omega)}{\text{cos.}\,\Phi\,\text{cos.}\,\omega}\cdot\frac{1}{\text{cos.}\,\omega}\cdot\frac{d\Phi}{\text{cos.}\,\Phi^2}\ \text{siue}$$

$$d\omega=nd\Phi\,\text{sin.}\,(\Phi-\omega)=d\Phi-(d\Phi-d\omega)\ \text{ergo}$$

$$d\Phi=\frac{d\Phi-d\omega}{1-n\,\text{sin.}\,(\Phi-\omega)}.$$

Posito ergo $\Phi-\omega=\psi$ erit

$$\Phi=\int\frac{d\psi}{1-n\,\text{sin.}\,\psi}\ \text{et}\ \omega=\int\frac{d\psi}{1-n\,\text{sin.}\,\psi}-\psi$$

hinc

hinc ob $p = \text{tang.}\,\Phi$; $u = \text{tang.}\,\omega$, obtinetur

$$\frac{dx}{x} = \frac{du \,\text{cof.}\,\Phi \,\text{cof.}\,\omega}{\text{fin.}\,\psi} = \frac{d\omega \,\text{cof.}\,\Phi}{\text{fin.}\,\Phi \,\text{cof.}\,\omega} = \frac{n\, d\psi \,\text{cof.}\,\Phi}{\text{cof.}\,\omega\,(1 - n\,\text{fin.}\,\psi)}.$$

Casu $n = 1$, fit

$$d\Phi = \frac{d\psi}{1 - \text{fin.}\,\psi} = \frac{d\psi\,(1 + \text{fin.}\,\psi)}{\text{cof.}\,\psi^2} \text{ ergo}$$

$$\Phi = \text{tang.}\,\psi + \frac{1}{\text{cof.}\,\psi} = \frac{1 + \text{fin.}\,\psi}{\text{cof.}\,\psi} + \alpha;\ \omega = \frac{1 + \text{fin.}\,\psi}{\text{cof.}\,\psi} + \alpha - \psi$$

$$\text{et } \frac{dx}{x} = \frac{d\Phi \,\text{cof.}\,\Phi}{\text{cof.}\,\omega} = \frac{d\Phi \,\text{cof.}\,\Phi}{\text{cof.}\,\Phi \,\text{cof.}\,\psi + \text{fin.}\,\Phi \,\text{fin.}\,\psi}.$$

Cum autem sit $\Phi - \alpha = \sqrt{\frac{1 + \text{fin.}\,\psi}{1 - \text{fin.}\,\psi}}$ erit

$$\text{fin.}\,\psi = \frac{(\Phi - \alpha)^2 - 1}{(\Phi - \alpha)^2 + 1},\ \text{cof.}\,\psi = \frac{2(\Phi - \alpha)}{(\Phi - \alpha)^2 + 1}.$$

$$\text{Ergo } \frac{dx}{x} = \frac{d\Phi \,\text{cof.}\,\Phi\,((\Phi - \alpha)^2 + 1)}{2(\Phi - \alpha)\,\text{cof.}\,\Phi + (\Phi - \alpha)^2\,\text{fin.}\,\Phi - \text{fin.}\,\Phi}.$$

Coroll. 1.

807. Si casu $n = 1$ sumatur constans α infinita, erit $\text{fin.}\,\psi = 1$, hinc $\psi = 90^\circ$ et $\omega = \Phi - 90^\circ$; tum vero: $\frac{dx}{x} = \frac{d\Phi \,\text{cof.}\,\Phi}{\text{fin.}\,\Phi}$, ideoque $x = a\,\text{fin.}\,\Phi$, et $u = -\text{cot.}\,\Phi$, ergo $y = -a\,\text{cof.}\,\Phi$, et $xx + yy = aa$.

Coroll. 2.

808. Eodem autem casu $n = 1$, quo constans α non sumitur infinita, fractionis cui $\frac{dx}{x}$ aequatur, numerator commode est differentiale denominatoris; vnde fit

$$x = a((\Phi - \alpha)^2\,\text{fin.}\,\Phi - \text{fin.}\,\Phi + 2(\Phi - \alpha)\,\text{cof.}\,\Phi).$$

 Tum

Tum vero eſt

$$\omega = \Phi - \text{Ang. tang.}\ \frac{(\Phi-\alpha)^2 - 1}{2(\Phi-\alpha)},\ \text{ideoque}$$

$$u = \frac{y}{x} = \text{tang.}\,\omega = \frac{\text{tang.}\,\Phi - \frac{(\Phi-\alpha)^2+1}{2(\Phi-\alpha)}}{1 + \frac{(\Phi-\alpha)^2-1}{2(\Phi-\alpha)}\,\text{tang.}\,\Phi}\ \text{ſeu}$$

$$\frac{y}{x} = \frac{2(\Phi-\alpha)\,\text{ſin.}\,\Phi - (\Phi-\alpha)^2\,\text{coſ.}\,\Phi + \text{coſ.}\,\Phi}{(\Phi-\alpha)^2\,\text{ſin.}\,\Phi - \text{ſin.}\,\Phi + 2(\Phi-\alpha)\,\text{coſ.}\,\Phi}\ \text{conſequenter}$$

$$y = -a((\Phi-\alpha)^2\,\text{coſ.}\,\Phi - \text{coſ.}\,\Phi - 2(\Phi-\alpha)\,\text{ſin}\,\Phi)\ \text{et}$$

$$\sqrt{(xx+yy)} = ((\Phi-\alpha)^2 + 1).$$

Scholion 1.

809. In genere etiam integrationem abſoluere licet. Cum enim ſit

$$d\Phi = \frac{d\psi}{1 - n\,\text{ſin.}\,\psi}\ \text{et}\ \frac{dx}{x} = \frac{n\,d\Phi\,\text{coſ.}\,\Phi}{\text{coſ.}\,\omega},\ \text{erit}$$

$$\Phi + \alpha = \frac{1}{\sqrt{(1-nn)}}\,\text{Ang. coſ.}\ \frac{n - \text{ſin.}\,\psi}{1 - n\,\text{ſin.}\,\psi}$$

vnde poſito

$$(\Phi+\alpha)\sqrt{(1-nn)} = \theta\ \text{erit}\ \text{coſ.}\,\theta = \frac{n - \text{ſin.}\,\psi}{1 - n\,\text{ſin.}\,\psi}\ \text{hincque}$$

$$\text{ſin.}\,\psi = \frac{n - \text{coſ.}\,\theta}{1 - n\,\text{coſ.}\,\theta}\ \text{et}\ \text{coſ.}\,\psi = \frac{\text{ſin.}\,\theta.\sqrt{(1-nn)}}{1 - n\,\text{coſ.}\,\theta}.$$

At ob $\omega = \Phi - \psi$, habebitur

$$\frac{dx}{x} = \frac{n\,d\Phi\,\text{coſ.}\,\Phi(1 - n\,\text{coſ.}\,\theta)}{\text{coſ.}\,\Phi\,\text{ſin.}\,\theta\sqrt{(1-nn)} + \text{ſin.}\,\Phi(n - \text{coſ.}\,\theta)}.$$

Iam cum ſit $d\theta = d\Phi\sqrt{(1-nn)}$, differentiale huius denominatoris eſt

$$-d\Phi\,\text{ſin.}\,\Phi\,\text{ſin.}\,\theta\sqrt{(1-nn)} + d\Phi\,\text{coſ.}\,\Phi\,\text{coſ.}\,\theta(1-nn) + n\,d\Phi\,\text{coſ.}\,\Phi$$
$$-d\Phi\,\text{coſ.}\,\Phi\,\text{coſ.}\,\theta + d\Phi\,\text{ſin.}\,\Phi\,\text{ſin.}\,\theta\sqrt{(1-nn)}$$

quod

quod redit ad $n\,d\Phi\,\mathrm{cof.}\,\Phi(1-n\,\mathrm{cof.}\,\theta)$ ipfum fcilicet numeratorem. Ita vt fit

$$x=a(\mathrm{cof}\,\Phi\,\mathrm{fin.}\,\theta\,\sqrt{(1-nn)}+\mathrm{fin.}\,\Phi\,(n-\mathrm{cof.}\,\theta))$$

feu $x=a\,\mathrm{cof.}\,\omega(1-n\,\mathrm{cof.}\,\theta)$, ideoque

$$y=ux=a\,\mathrm{fin.}\,\omega(1-n\,\mathrm{cof}\,\theta).$$

Affumto ergo angulo θ, quaeratur angulus ψ vt fit

$$\mathrm{fin.}\,\psi=\frac{n-\mathrm{cof.}\,\theta}{1-n\,\mathrm{cof.}\,\theta}\quad\text{et}\quad\mathrm{cof.}\,\psi=\frac{\mathrm{fin.}\,\theta}{1-n\,\mathrm{cof.}\,\theta}\sqrt{(1-nn)}$$

tum vero fiat

$$\omega=\frac{\theta}{\sqrt{(1-nn)}}-\alpha-\psi$$

eritque integrale completum

$$x=a(1-n\,\mathrm{cof.}\,\theta)\,\mathrm{cof.}\,\omega\quad\text{et}\quad y=a(1-n\,\mathrm{cof.}\,\theta)\,\mathrm{fin.}\,\omega.$$

Scholion 2.

810. At fi numerus n fit vnitate maior, haec integratio fit imaginaria, quod incommodum vt tollatur, notandum eft aequationis $d\Phi=\frac{d\psi}{1-n\,\mathrm{fin.}\,\psi}$ integrale effe:

$$\Phi+\alpha=\frac{1}{\sqrt{(nn-1)}}\,l\,\frac{\sqrt{(n-1)(1+\mathrm{fin.}\,\psi)}+\sqrt{(n+1)(1-\mathrm{fin.}\,\psi)}}{\sqrt{(n-1)(1+\mathrm{fin.}\,\psi)}-\sqrt{(n+1)(1-\mathrm{fin.}\,\psi)}}.$$

Quare fi ponatur $(\Phi+\alpha)\sqrt{(nn-1)}=\theta$, vt fit

$$d\theta=d\Phi\sqrt{(nn-1)}\quad\text{et}\quad\omega=\Phi-\psi=\frac{\theta}{\sqrt{(nn-1)}}-\alpha-\psi$$

erit

$$\frac{e^\theta+1}{e^\theta-1}=\frac{\sqrt{(n-1)}(1+\mathrm{fin.}\,\psi)}{\sqrt{(n+1)}(1-\mathrm{fin.}\,\psi)}=\frac{(n-1)(1+\mathrm{fin.}\,\psi)}{\mathrm{cof.}\,\psi\sqrt{(nn-1)}}$$

 vnde

vnde reperitur:

$$\text{fin.}\,\psi = \frac{e^\theta + 2n + e^{-\theta}}{ne^\theta + 2 + ne^{-\theta}} \quad \text{et} \quad \text{cof.}\,\psi = \frac{(e^\theta - e^{-\theta})\sqrt{(nn-1)}}{ne^\theta + 2 + ne^{-\theta}}$$

ita vt ex angulo θ definiantur anguli Φ, ψ et ω. Cum iam fit

$$\frac{dx}{x} = \frac{n\,d\Phi\,\text{cof.}\,\Phi}{\text{cof.}\,\omega} = \frac{n\,d\Phi\,\text{cof.}\,\Phi}{\text{cof.}\,\Phi\,\text{cof.}\,\psi + \text{fin.}\,\Phi\,\text{fin.}\,\psi} \quad \text{erit}$$

$$\frac{dx}{x} = \frac{n\,d\Phi\,\text{cof.}\,\Phi(ne^\theta + 2 + ne^{-\theta})}{\text{cof.}\,\Phi(e^\theta - e^{-\theta})\sqrt{(nn-1)} + \text{fin.}\,\Phi(e^\theta + 2n + e^{-\theta})}$$

vbi iterum commode euenit, vt numerator fit ipfum differentiale denominatoris, quemadmodum differentiationem inftituenti mox patebit. Hinc ergo erit

$$x = a(\text{cof.}\,\Phi(e^\theta - e^{-\theta})\sqrt{(nn-1)} + \text{fin.}\,\Phi(e^\theta + 2n + e^{-\theta})) \quad \text{feu}$$

$$x = a\,\text{cof.}\,\omega(ne^\theta + 2 + ne^{-\theta}) \quad \text{et ob} \quad u = \tfrac{y}{x} = \text{tang.}\,\omega \quad \text{fit}$$

$$y = a\,\text{fin.}\,\omega(ne^\theta + 2 + ne^{-\theta}).$$

Quocirca ex angulo θ primo quaeratur angulus ψ, vt fit

$$\text{fin.}\,\psi = \frac{e^\theta + 2n + e^{-\theta}}{ne^\theta + 2 + ne^{-\theta}} \quad \text{et} \quad \text{cof.}\,\psi = \frac{(e^\theta - e^{-\theta})\sqrt{(nn-1)}}{ne^\theta + 2 + ne^{-\theta}}$$

quo inuento capiatur angulus $\omega = \frac{\theta}{\sqrt{(nn-1)}} - \alpha - \psi$, ac formulae illae pro x et y inuentae dabunt integrale completum ob duas conftantes a et α introductas.

Scholion 3.

Scholion 3.

811. Cum hic praecipua pars integrationis fortuito successisse videatur, operae praetium erit in eius causam inquirere, num forte ratio integrandi clarius perspici queat. Cum igitur sit:

$$\Phi = \psi + \omega \text{ et } d\Phi = \frac{d\Phi}{1 - n\,\text{fin.}\,\psi}\text{, hincque}$$

$$d\omega = \frac{n\,d\,\psi\,\text{fin.}\,\psi}{1 - n\,\text{fin.}\,\psi} = n\,d\Phi\,\text{fin.}\,\psi \text{ aequatio}$$

$$\frac{dx}{x} = \frac{n\,d\,\Phi\,\text{cof.}\,\Phi}{\text{cof.}\,\omega} \text{ ob } \text{cof.}\,\Phi = \text{cof.}\,\psi\,\text{cof.}\,\omega - \text{fin.}\,\psi\,\text{fin.}\,\omega$$

in tres has resoluitur

$$\frac{dx}{x} = n\,d\Phi\,\text{cof.}\,\psi - \frac{n\,d\,\Phi\,\text{fin.}\,\psi\,\text{fin.}\,\omega}{\text{cof.}\,\omega} \text{ quae ob}$$

$$d\Phi = \frac{d\,\psi}{1 - n\,\text{fin.}\,\psi} \text{ et } n\,d\Phi\,\text{fin.}\,\psi = d\omega$$

induit hanc formam integrabilem:

$$\frac{dx}{x} = \frac{n\,d\,\psi\,\text{cof.}\,\psi}{1 - n\,\text{fin.}\,\psi} - \frac{d\omega\,\text{fin.}\,\omega}{\text{cof.}\,\omega}$$

ex qua elicitur:

$$x = \frac{a\,\text{cof.}\,\omega}{1 - n\,\text{fin.}\,\psi} \text{ et } y = \frac{a\,\text{fin.}\,\omega}{1 - n\,\text{fin.}\,\psi} \text{ ob } y = ux = x\,\text{tang.}\,\omega.$$

En ergo in genere aequationis nostrae hanc integrationem. Anguli ω et ψ hanc inter se teneant relationem; vt sit:

$$d\omega = \frac{n\,d\,\psi\,\text{fin.}\,\psi}{1 - n\,\text{fin.}\,\psi}\text{, tum vero erit}$$

$$x = \frac{a\,\text{cof.}\,\omega}{1 - n\,\text{fin.}\,\psi} \text{ et } y = \frac{a\,\text{fin.}\,\omega}{1 - n\,\text{fin.}\,\psi}.$$

Quodsi

Quodsi ergo ponamus $\sqrt{(xx+yy)}=z$, vt sit

$$x=z\cos.\omega \text{ et } y=z\sin.\omega \text{ erit}$$

$$z=\frac{a}{1-n\sin.\psi} \text{ et } \sin.\psi=\frac{z-a}{nz}, \text{ hinc}$$

$$d\omega=\frac{(z-a)d\psi}{a}. \text{ At fit}$$

$$d\psi=\frac{adz}{z\sqrt{(nnzz-(z-a)^2)}} \text{ ergo } d\omega=\frac{(z-a)dz}{z\sqrt{(nnzz-(z-a)^2)}}$$

vnde angulus ω per z definitur. Ad irrationalitatem tollendam, si ponamus $\sqrt{(nnzz-(z-a)^2)}=s(nz+z-a)$ fit

$$z=\frac{a(ss+1)}{(n+1)ss-n+1} \text{ et } d\omega=\frac{2nds(ss-1)}{(ss+1)((n+1)ss-(n-1))},$$

seu $d\omega=\frac{2ds}{ss+1}-\frac{2ds}{(n+1)ss-n+1}$

quae integratio est manifesta.

Problema 99.

812. Si aequatio differentio-differentialis tum demum fiat homogenea, si alteri variabili y tribuantur n dimensiones, eius integrationem ad aequationem differentialem primi gradus reducere.

Solutio.

Posito $dy=pdx$ et $dp=qdx$, vt oriatur aequatio inter quaternas quantitates finitas x, y, p et q, quae quomodo ratione homogeneitatis futura sit comparata, videamus. Primo ergo cum pro x vnam dimensionem numerando, variabilis y habeat n dimensiones, quantitati $p=\frac{dy}{dx}$ tribuendae sunt $n-1$ dimen-

dimensiones et quantitati $q=\frac{dp}{dx}$, $n-2$ dimensiones. Quocirca ponamus:

$$y=x^n u\,,\; p=x^{n-1}t \text{ et } q=x^{n-2}v$$

et ob $dy=pdx$ et $dp=qdx$ habebimus:

$$xdu+nudx=tdx \text{ et } xdt+(n-1)tdx=vdx$$

vnde colligimus:

$$\frac{dx}{x}=\frac{du}{t-nu}=\frac{dt}{v-(n-1)t}$$

ideoque $du(v-(n-1)t)=dt(t-nu)$.

At factis superioribus substitutionibus in aequatione inter x, y, p et q per hypothesin variabilis x ex calculo extruditur, ita vt prodeat aequatio inter tres tantum variabiles u, t et v, ex qua litteram v per binas t et u definire licebit. Quo valore substituto habebitur aequatio differentialis primi gradus inter binas variabiles u et t, ex qua t per u determinari queat, ope aequationis $\frac{dx}{x}=\frac{du}{t-nu}$, definietur x per u hincque ob $u=\frac{y}{x^n}$ obtinebitur aequatio integralis inter x et y, eaque ob duplicem integrationem completa.

Coroll. 1.

813. Aequationum ergo inter x, y, p et q hoc modo tractabilium hoc est criterium, vt posito $y=x^n u$, $p=x^{n-1}t$ et $q=x^{n-2}v$, exponens n eius-

modi determinationem patiatur, vt variabilis x prorſus ex calculo per diuiſionem egrediatur.

Coroll. 2.

814. Si ſit $n = 0$, aequatio ita eſt comparata, vt tribuendo ipſi y eiusque differentialibus nullam dimenſionem fiat homogenea. Hoc ſcilicet caſu ſola variabilis x cum ſuis differentialibus dimenſiones conſtituere cenſetur.

Coroll. 3.

815. Contra vero ſi dimenſiones ex ſola variabili y aeſtimentur, ita vt ea cum ſuis differentialibus dy et ddy vbique eundem dimenſionum numerum conſtituat, exponens n fiet infinitus.

Scholion.

816. Si ſola variabilis x cum ſuis differentialibus vbique eundem dimenſionum numerum complet, ob $n = 0$ fit $u = y$; et in aequatione inter x, y, p et q ſtatui conueniet $p = \frac{t}{x}$ et $q = \frac{v}{xx}$, quo facto variabilis x ex calculo deturbabitur, prodibitque aequatio inter y, t et v cuius ope aequatio differentialis $dy(v+t) = t\,dt$, ad duas tantum variabiles reducetur, qua reſoluta, erit $\frac{dx}{x} = \frac{dy}{t}$, vbi cum t detur per y integratio nullam habet difficultatem. Verum altero caſu, quo variabilis y ſola cum ſuis differentialibus pares vbique dimenſiones habet,

habet, ideoque exponens n infinitus capi deberet, resolutio alio modo institui debet, quem mox docebimus; nisi forte permutando variabiles x et y casum ad praecedentem reducere lubuerit.

Exemplum 1.

817. *Sumto elemento* dx *constante, si proponatur haec aequatio* $xxddy = \alpha y dx^2 + \beta x dx dy$, *eius integrale inuenire.*

Perpendatur hic ista conditio, qua sola variabilis x cum suo differentiali dx vbique duas constituit dimensiones, eritque $n = 0$. Cum ergo posito $dy = p dx$ et $dp = q dx$ habeamus $qxx = \alpha y + \beta px$, statuamus $p = \frac{t}{x}$ et $q = \frac{v}{xx}$, fietque $v = \alpha y + \beta t$, vnde adipiscimur istam aequationem differentialem

$$\alpha y dy + (\beta + 1) t dy = t dt$$

ob cuius homogeneitatem faciamus $t = yz$, eritque

$$\alpha dy + (\beta + 1) z dy = yz dz + zz dy$$

seu $\frac{dy}{y} = \frac{z dz}{\alpha + (\beta + 1) z - zz}$; sit

$$\alpha + (\beta + 1) z - zz = (f + z)(g - z) \text{ vt sit}$$

$$\alpha = fg \text{ et } \beta + 1 = g - f, \text{ reperieturque}$$

$$\frac{dy}{y} = \frac{\frac{-f}{f+g} dz}{f+z} + \frac{\frac{g}{f+g} dz}{g-z}$$

 vnde

vnde colligitur integrando

$$ly = C - \frac{f}{f+g} l(f+z) - \frac{g}{f+g} l(g-z)$$

seu $y(f+z)^{\frac{f}{f+g}}(g-z)^{\frac{g}{f+g}} = a.$

Tum vero est

$$\frac{dx}{x} = \frac{dy}{yz} = \frac{dz}{(f+z)(g-z)} \text{ seu}$$

$$\frac{dx}{x} = \frac{1}{f+g} \cdot \frac{dz}{f+z} + \frac{1}{f+g} \cdot \frac{dz}{g-z} \text{ hinc}$$

$$x = b\left(\frac{f+z}{g-z}\right)^{\frac{1}{f+g}} \text{ seu } \frac{f+z}{g-z} = \left(\frac{x}{b}\right)^{f+g}.$$

Vnde cum sit $z = \frac{g x^{f+g} - f b^{f+g}}{b^{f+g} + x^{f+g}}$ erit hoc valore ibi substituto:

$$(f+g) b^g x^f y = a(b^{f+g} + x^{f+g}) \text{ seu posito } \frac{a}{f+g} = c$$

$$y = c\left(\frac{b^f}{x^f} + \frac{x^g}{b^g}\right) \text{ est vero}$$

$$g - f = \beta + 1 \text{ et } f + g = \sqrt{((\beta+1)^2 + 4\alpha)}.$$

Corollarium.

818. Quoniam in aequatione proposita etiam ambae variabiles x et y simul vbique totidem dimensiones habent, eam etiam secundum praecepta praecedentis problematis tractare licet.

Exem-

Exemplum 2.

819. *Posito* dx *constante si aequatio differentio-differentialis duobus tantum terminis constet, vt sit huiusmodi* $ddy = cx^{\alpha} y^{\beta} dx^{2-\gamma} dy^{\gamma}$ *eius integrale investigare.*

Posito $dy = pdx$ et $dp = qdx$ habebitur haec forma $q = cx^{\alpha} y^{\beta} p^{\gamma}$, vbi exponentem n ita definire licet vt posito $y = x^n u$; $p = x^{n-1} t$ et $q = x^{n-2} v$, variabilis x per diuisionem tolli possit, capi enim oportet $\alpha + \beta n + \gamma(n-1) - n + 2 = 0$ seu $n = \frac{-\alpha + \gamma - 2}{\beta + \gamma - 1}$; tumque erit $v = cu^{\beta} t^{\gamma}$. Aequatio ergo differentialis primi gradus resoluenda erit

$$cu^{\beta} t^{\gamma} du - (n-1) t du = t dt - nu dt$$

ex qua cum variabilis t per u fuerit determinata integrari oportet hanc formulam $\frac{dx}{x} = \frac{du}{t - nu}$ quo facto ob $u = \frac{y}{x^n}$, obtinebitur aequatio integralis quaesita inter x et y.

Casus tantum $\beta + \gamma = 1$, quo n fit infinitus, peculiarem postulat tractationem infra exponendam, nisi forte simul sit $\gamma = \alpha + 2$, tum enim exponens n prorsus arbitrio nostro relinquitur, at aequatio erit homogenea.

Exemplum 3.

820. *Sumto elemento* dx *constante*, *si proponatur haec aequatio*:

$$x^4 ddy = x^2 dxdy + 2xydxdy - 4yydx^2,$$

eius integrale inuenire.

Hic euidens est si ipsi y eiusque differentialibus dy et ddy binae dimensiones, ipsi x vero et dx singulae tribuantur, in omnibus terminis obtineri 6 dimensiones. Quare cum posito $dy = pdx$ et $dp = qdx$ habeamus hanc aequationem:

$x^4 q = x^2 p + 2xyp - 4yy$, faciamus

$y = x^2 u$, $p = xt$ et $q = v$ prodibitque

$v = t + 2ut - 4uu.$

At ob $n = 2$ aequatio differentialis nostra erit

$du(v-t) = dt(t-2u)$, quae abit in

$2udu(t-2u) = dt(t-2u)$

vnde deducimus vel $t = 2u$ vel $t = uu + c$, quos binos casus seorsim euoluamus.

1) Si $t = 2u$, ob $\frac{dx}{x} = \frac{du}{t-2u}$ fit $du = 0$, ideoque $u = C$, ac propterea $y = Cxx$, quod est integrale particulare, aequationi propositae vtique satisfaciens.

2) Sit $t = uu + c$, erit $\frac{dx}{x} = \frac{du}{uu-2u+c}$, vbi tres casus sunt considerandi:

Primo si constans $c = 1$ erit

$l\frac{x}{a} = \frac{1}{1-u} = \frac{xx}{xx-y}$ seu $xx = (xx-y)l\frac{x}{a}$.

Secundo

Secundo si constans $c = 1 - ff$ erit $\frac{dx}{x} = \frac{du}{(u-1)^2 - ff}$ hincque

$$lx = \frac{-1}{2f} l \frac{f+u-1}{f-u+1} + C,$$

ergo ob $u = \frac{y}{xx}$ erit

$$x = a\left(\frac{(f+1)xx - y}{(f-1)xx + y}\right)^{\frac{1}{2f}}.$$

Tertio si constans $c = 1 + ff$ ideoque $\frac{dx}{x} = \frac{du}{(u-1)^2 + ff}$ quae integrata dat

$$l\frac{x}{a} = \frac{1}{f}\,\text{Ang.tang.}\frac{u-1}{f}, \text{ seu } \frac{u-1}{f} = \frac{y - xx}{fxx} = \text{tang.} f\, l\frac{x}{a}.$$

Pro ratione ergo constantis arbitrariae c integratio vel algebraice succedit, vel a logarithmis vel ab angulis pendet, vnde forma generali exprimi nequit.

Scholion.

821. Integrale autem particulare primo inventum $y = Cxx$ in nulla harum formarum, quibus integrale completum constituitur, contineri deprehenditur: nihilo vero minus satisfacit aequationi differentio-differentiali propositae. Hoc ergo exemplo magis illustrantur ea, quae supra circa hoc paradoxon sumus commentati, quod interdum aequationi differentiali satisfaciat aequatio finita, quae in integrali completo minime contineatur. Videmus igitur hoc idem paradoxon etiam in aequationibus differentio-differentialibus locum habere. Vtrum autem illa aequatio $y = Cxx$ inter integralia sit admittenda, alia

alia est quaestio, quae nondum penitus videtur confecta; hic quidem ipsa aequatio proposita quasi factores habens est censenda, ex quorum altero illa aequatio $y = Cxx$ nascatur, verum multum abest, vt in hac explicatione acquiescere queamus. Quin potius ipsa illa quaestio, siue geometrica fuerit siue alius disciplinae, cuius solutio ad huiusmodi aequationem perduxerit, accurate perpendi debere videtur; vbi plerumque haud difficulter iudicari potest, vtrum quicquid aequationi differentiali satisfaciat, id etiam ipsi quaestioni conueniat nec ne? Veluti si descensus grauis ex altitudine $= a$ labentis definiri debeat, et altitudo qua iam a terra distat sit x, erit ibi celeritas vt $\sqrt{(a-x)}$ et elementum temporis $dt = \frac{-dx}{\sqrt{(a-x)}}$. Hic quidem euidens est isti aequationi differentiali satisfieri ponendo $x = a$, ita vt tempus t maneat indefinitum, quod tamen quaestioni neutiquam conuenit, quae nonnisi vero integrali $t = 2\sqrt{(a-x)}$ resoluitur.

Problema 100.

822. Si in aequatione differentio-differentiali variabilis y cum suis differentialibus dy et ddy vbique eundem dimensionum numerum adimpleat, eius integrationem ad aequationem differentialem primi gradus reducere.

Solutio.

Solutio.

Posito $dy = pdx$ et $dp = qdx$ aequatio ita erit comparata vt in ea ternae variabiles y, p, q vbique eundem dimensionum numerum obtineant, altera variabili x in computum dimensionum prorsus non ingrediente. Quare si statuatur $p = uy$ et $q = vy$, in omnibus terminis inerit eadem ipsius y potestas, qua per diuisionem sublata habebitur aequatio inter ternas tantum variabiles x, u et v, ex qua vnam per binas reliquas definire licebit, ita vt v aequetur functioni cuipiam ipsarum x et u. Iam ob $p = uy$ erit $dy = uydx$, et ob $dp = qdx$ fiet $udy + ydu\,vydx$, vnde sequitur

$$\frac{dy}{y} = udx \text{ et } \frac{dy}{y} = \frac{vdx - du}{u}$$

ideoque $du + uudx = vdx$, quae aequatio differentialis duas tantum variabiles x et u complectitur. Quam ergo si integrare liceat, vt relatio inter x et u, inde innotescat, superest, vt formulae udx integrale inuestigetur, quo inuento erit $ly = \int udx$, sicque aequatio orietur integralis inter x et y, quae ob duplicem integrationem peractam duas constantes arbitrarias inuoluet, ideoque integrale completum exhibebit.

Coroll. 1.

823. Huiusmodi ergo aequationum integratio reducitur ad huiusmodi aequationem differentialem $du + uudx = vdx$, cuius resolutio si succedat, simul

 illarum

illarum integratio habetur, cum formulae udx integratio difficultate careat.

Coroll. 2.

824. Cum sit $\frac{dy}{y}=udx$ erit $y=e^{\int udx}$, qua substitutione aequatio differentio-differentialis proposita statim reducitur ad aequationem differentialem primi gradus, erit enim:

$$\frac{dy}{dx}=p=e^{\int udx}u \text{ et } \frac{dp}{dx}=q=e^{\ udx}(du+uudx)$$

ac tum formula exponentialis sponte ex aequatione egreditur.

Coroll. 3.

825. Vicissim etiam proposita aequatione differentiali primi gradus $du+uudx=vdx$, in qua v sit functio quaecunque ipsarum x et u, ea posito $u=\frac{dy}{ydx}$, in eiusmodi aequationem differentio-differentialem transformatur, in qua variabilis y cum suis differentialibus dy et ddy vbique eundem dimensionum numerum constituat.

Scholion 1.

826. Haec reductio aequationum differentialium primi gradus ad gradum secundum legibus analyseos aduersari videtur, interim tamen subinde vsu non caret; quodsi enim alia methodo huiusmodi aequationes differentio-differentiales tractare liceat;

liceat; dum earum integralia vel per feries exhibentur vel finite, fimul integralia aequationum differentialium primi gradus innotefcunt, quorum ratio plerumque aliunde vix perfpicitur. In fequentibus autem videbimus, eiusmodi aequationes differentio-differentiales in quibus variabilis altera y vnam dimenfionem non fuperat, per feries commode integrari poffe, atque adeo interdum has feries abrumpi, ita vt integrale finita expreffione exhibeatur. Caeterum propofita huiusmodi aequatione differentiali primi gradus $du + uudx = vdx$ fubftitutio $u = \frac{dy}{ydx}$, eo magis eft notatu digna, quod fumto elemento dx conftante fiat $du = \frac{ddy}{ydx} - \frac{dy^2}{yydx}$ ideoque $du + uudx = \frac{ddy}{ydx}$, ita vt duo termini hoc modo in vnum coalefcant.

Scholion 2.

827. Cafus hic imprimis notaffe iuuabit, quibus aequatio $du + uudx = vdx$ integrationem admittit. Hunc in finem fit $du = Vdx$ forma generalis aequationum refolubilium, et V certa functio ipfarum x et u, ac manifeftum eft fi fuerit $v = uu + V$ integrationem fuccedere. Primum ergo hoc eueniet fi fit $V = \frac{X}{U}$, denotante X functionem ipfius, x et U ipfius [illegible] Secundo fi V fit functio homogenea nullius dimenfionis ipfarum x et u. Tertio fi denotantibus X et Ξ functiones quascunque ipfius x fuerit $V = Xu + \Xi u^n$. Quarto fi denotantibus P et Q

 functio-

functiones quascunque ipsius u, fuerit $V=\frac{1}{Px+Qx^n}$. Similique modo ex aliis formis integrabilibus alii casus concludentur.

Exemplum 1.

828. *Sumto elemento* dx *constante, si proponatur haec aequatio*:

$$\alpha yddy+\beta dy^2=\frac{\gamma dxdy}{\sqrt{(aa+xx)}}$$

eius integrale inuenire.

Posito $dy=pdx$ et $dp=qdx$, prodit:

$$\alpha yq+\beta pp=\frac{\gamma p}{\sqrt{(aa+xx)}}$$

quae facto $p=uy$ et $q=vy$ abit in hanc

$$\alpha v+\beta uu=\frac{u}{\sqrt{(aa+xx)}} \text{ seu } v=\frac{u}{\alpha\sqrt{(aa+xx)}}-\frac{\beta uu}{\alpha}$$

vnde hanc aequationem resolui oportet

$$du+uudx=\frac{udx}{\alpha\sqrt{(aa+xx)}}-\frac{\beta uudx}{\alpha}.$$

Statuatur $u=\frac{1}{s}$ fietque

$$+ds+\frac{sdx}{\alpha\sqrt{(aa+xx)}}=(1+\frac{\beta}{\alpha})dx$$

quae per $(x+\sqrt{(aa+xx)})^{\frac{1}{\alpha}}$ multiplicata et integrata dat:

$$s(x+\sqrt{(aa+xx)})^{\frac{1}{\alpha}}=(1+\frac{\beta}{\alpha})\int dx(x+\sqrt{(aa+xx)})^{\frac{1}{\alpha}}.$$

Fiat $x+\sqrt{(aa+xx)}=t^\alpha$, erit $aa=t^{2\alpha}-2t^\alpha x$, hinc

hinc

$$x=\frac{t^{2\alpha}-aa}{2t^{\alpha}}=\tfrac{1}{2}t^{\alpha}-\tfrac{1}{2}aat^{-\alpha} \text{ et } dx=\tfrac{\alpha}{2}dt(t^{\alpha-1}+aat^{-\alpha-1})$$

ita vt sit

$$st=(1+\tfrac{\beta}{\alpha})\int\tfrac{\alpha}{2}dt(t^{\alpha}+aat^{-\alpha}) \text{ seu}$$

$$st=C+\frac{\alpha+\beta}{2}\left(\frac{t^{\alpha+1}}{\alpha+1}+\frac{aat^{1-\alpha}}{1-\alpha}\right).$$

Porro est $\frac{dy}{y}=udx=\frac{dx}{s}$; at ex aequatione differentiali est

$$(1+\tfrac{\beta}{\alpha})\tfrac{dx}{s}=\tfrac{ds}{s}+\tfrac{dx}{\alpha\sqrt{(aa+xx)}} \text{ hincque}$$

$$(1+\tfrac{\beta}{\alpha})ly=ls+\tfrac{1}{\alpha}l(x+\sqrt{(aa+xx)})=lst+D$$

ergo $y=B(st)^{\frac{\alpha}{\alpha+\beta}}$. Quare sumto $C=\frac{\alpha+\beta}{2}A$ habebitur:

$$y=B\left(A+\frac{t^{\alpha+1}}{\alpha+1}+\frac{aat^{1-\alpha}}{1-\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}$$

existente

$$x=\tfrac{1}{2}(t^{\alpha}-aat^{-\alpha}) \text{ vel } t=(x+\sqrt{(aa+xx)})^{\frac{1}{\alpha}}$$

ita vt aequatio inter x et y sit

$$Cy^{\frac{\alpha+\beta}{\alpha}}=A+\tfrac{1}{\alpha+1}(x+\sqrt{(aa+xx)})^{\frac{\alpha+1}{\alpha}}+\tfrac{aa}{1-\alpha}(x+\sqrt{(aa+xx)})^{\frac{1-\alpha}{\alpha}}.$$

Scholion.

829. Hoc idem exemplum ita eſt comparatum, vt alia ratione facillime reſolui poſſit; aequatio enim $\alpha yq + \beta pp = \frac{yp}{\sqrt{(aa+xx)}}$, ſi per $\frac{dx}{yp}$ multiplicetur ob $q\,dx = dp$ et $p\,dx = dy$, abit in $\frac{\alpha dp}{p} + \frac{\beta dy}{y} = \frac{dx}{\sqrt{(aa+xx)}}$, cuius ſinguli termini ſunt integrabiles. Prodit ergo

$$p^\alpha y^\beta = C^\alpha (x + \sqrt{(aa+xx)}) \text{ hincque}$$

$$y^{\frac{\beta}{\alpha}} dy = C dx (x + \sqrt{(aa+xx)})^{\frac{1}{\alpha}}$$

quae aequatio denuo integrata praebet integrale ante inuentum. Forma ergo generalis aequationum hoc modo reſolubilium eſt $P dp + Y dy + X dx = 0$, exiſtente P functione ipſius p, Y ipſius y et X ipſius x, quae noſtro more repraeſentatur per $Pq + Yp + X = 0$. Hinc ergo perſpicitur quomodo etiam aequationes differentio-differentiales ope idonei multiplicatoris ad integrationem perduci queant; quae methodus, cum in aequationibus differentialibus primi gradus inſignem vſum praeſtiterit, eo magis excolenda videtur, quod etiam ad aequationes differentiales altiorum graduum pateat, quod argumentum infra fuſius pertractare conabimur.

Exemplum 2.

830. *Sumto elemento* dx *conſtante, ſi proponatur haec aequatio* $xyddy = ydxdy + xdy^2 + \frac{bxdy^2}{\sqrt{(aa-xx)}}$, *eius integrale inuenire.*

Poſito

Posito $dy = pdx$ et $dp = qdx$ erit

$$xyq = yp + xpp + \frac{bxpp}{\sqrt{(aa-xx)}}$$

quae facto $p = uy$ et $q = vy$ abit in

$$xv = u + uux + \frac{bux}{\sqrt{(aa-xx)}}$$

vnde oritur haec aequatio differentialis

$$du + uudx = \frac{udx}{x} + uudx + \frac{buudx}{\sqrt{(aa-xx)}} \text{ seu}$$

$$\frac{xdu - udx}{uu} = \frac{bxdx}{\sqrt{(aa-xx)}} \text{ cuius integrale}$$

$$C - \frac{x}{u} = -b\sqrt{(aa-xx)} \text{ seu}$$

$$u = \frac{x}{C + b\sqrt{(aa-xx)}} \text{ ergo } \frac{dy}{y} = \frac{xdx}{C + b\sqrt{(aa-xx)}}.$$

Statuatur $\sqrt{(aa-xx)} = t$, vt sit $xdx = -tdt$ erit

$$\frac{dy}{y} = \frac{-tdt}{C+bt} = -\frac{dt}{b} + \frac{Cdt}{b(C+bt)} \text{ et}$$

$$ly = -\frac{t}{b} + \frac{C}{bb}l(C+bt) + lc$$

Sit $C = nbb$ erit

$$l\frac{y}{c} = -\frac{\sqrt{(aa-xx)}}{b} + nl\frac{nb + \sqrt{(aa-xx)}}{b}$$

vbi c et n sunt constantes arbitrariae.

CAPVT IV.

CAPVT IV.

DE AEQVATIONIBVS DIFFERENTIO-DIFFERENTIALIBVS IN QVIBVS ALTERA VARIABILIS VNICAM HABET DIMENSIONEM.

Problema 101.

831.

Sumto elemento dx constante si proponatur aequatio huius formae $ddy + Pdxdy + Qydx^2 = 0$, vbi P et Q sint functiones quaecunque ipsius x, eam ad aequationem differentialem primi gradus reuocare.

Solutio.

Ponendo $dy = pdx$ et $dp = qdx$ aequatio proposita induit hanc formam $q + Pp + Qy = 0$, in qua si methodo ante exposita statuamus $p = uy$ et $q = vy$, obtinebimus hanc inter x, u et v aequationem $v + Pu + Q = 0$, hincque $v = -Pu - Q$. Tum vero fit

$$dy = uydx \text{ et } udy + ydu = vydx$$

ita

ita vt sit

$$\frac{dy}{y} = u\,dx = \frac{v\,dx - du}{u}, \text{ ideoque}$$

$$du + uu\,dx + Pu\,dx + Q\,dx = 0$$

substituto pro v valore.

Qua aequatione resoluta erit $ly = \int u\,dx$.

Vel sine his substitutionibus statim in ipsa aequatione proposita ponamus $y = e^{\int u\,dx}$, vnde fit

$$dy = e^{\int u\,dx} u\,dx \text{ et } ddy = e^{\int u\,dx}(du\,dx + uu\,dx^2)$$

et cum facta substitutione quantitas exponentialis $e^{\int u\,dx}$ ex calculo tollatur, obtinebitur praecedens aequatio differentialis primi gradus:

$$du + uu\,dx + Pu\,dx + Q\,dx = 0$$

a cuius resolutione integratio aequationis differentio-differentialis propositae pendet.

Coroll. 1.

832. Haec aequatio differentialis primi gradus pluribus modis in alias formas sibi fere similes transmutari potest. Veluti si ponamus $u = Mz$; prodit

$$M\,dz + z(dM + PM\,dx) + MMzz\,dx + Q\,dx = 0$$

vbi pro M eiusmodi functionem ipsius x accipere licet, vt terminus ipsa litera z affectus euanescat, quod fit si

$$dM + MP\,dx = 0 \text{ seu } M = Ce^{-\int P\,dx}.$$

Coroll. 2.

833. Similis forma prodit ponendo $u = \frac{K}{z}$, fit enim

$$-\frac{Kdz}{zz} + \frac{dK}{z} + \frac{KKdx}{zz} + \frac{KPdx}{z} + Qdx = 0 \text{ seu}$$

$$Kdz - z(dK + KPdx) - Qzzdx - KKdx = 0$$

vbi secundus terminus sumendo $K = Ce^{-\int Pdx}$ pariter euanescit.

Coroll. 3.

834. Similis transformatio generalius instituitur ponendo $u = K + Mz$, prodit enim

$$dK + Mdz + zdM + KKdx + 2KMzdx + MMzzdx + KPdx + MPzdx + Qdx = 0$$

quae ordinata praebet:

$$Mdz + z(dM + 2KMdx + MPdx) + MMzzdx + dK + KKdx + KPdx + Qdx = 0$$

vnde secundus terminus tollitur sumendo

$$M = Ce^{-\int dx(2K + P)} \text{ seu } K = \frac{-dM - MPdx}{2Mdx}.$$

Coroll. 4.

835. Adhuc generalius similis forma oritur, si ponatur $u = \frac{K + Mz}{L + Nz}$, vnde prodit

$$dz(LM - KN) + LdK - KdL + z(LdM - MdL + NdK - KdN) + zz(NdM - MdN) + (K + Mz)^2dx + P(K + Mz)(L + Nz)dx + Q(L + Nz)^2dx = 0$$

quae

quae reducitur ad hanc formam:

$$\begin{aligned}0 = {} & dz(LM-KN)\\ & + z\left\{\begin{array}{l} LdM-MdL+NdK-KdN \\ +2KMdx+P(KN+LM)dx+2LNQdx \end{array}\right\}\\ & + zz(NdM-MdN+MMdx+MNPdx+NNQdx)\\ & + LdK-KdL+KKdx+KLPdx+LLQdx\end{aligned}$$

vbi pro H, L, M et N functiones eiusmodi ipsius x accipere licet, vt forma prodeat tractatu facillima.

Scholion.

836. Quoniam huiusmodi aequationes differentio-differentiales, in quibus variabilis y vnicam habet dimensionem frequentissime occurrere solent; merito geometrae tantopere in resoluenda aequatione

$$du + uudx + Pudx + Qdx = 0$$

studium et operam collocarunt, quae etiam forma generaliori ita repraesentari potest

$$dz + Pzdx + Rzzdx + Qdx = 0$$

cuius quidem casum eximium $dz + zzdx = ax^n dx$ olim Comes *Riccati* in haud spernendum Analyseos incrementum proposuerat. Inter transformationes autem huius casus praecipue notari meretur positio $x = t^{\frac{2}{n+2}}$ quae dat

$$dz + \frac{2}{n+2} zzt^{\frac{-n}{n+2}} dt = \frac{2a}{n+2} t^{\frac{n}{n+2}} dt$$

M 2 vnde

vnde ponendo $z = Ct^{\frac{n}{n+2}}v$ prodit

$$(n+2)Ct^{\frac{n}{n+2}}dv + Cnt^{\frac{-2}{n+2}}vdt + 2CCt^{\frac{n}{n+2}}vvdt = 2at^{\frac{n}{n+2}}dt$$

seu $(n+2)Cdv + \frac{nCvdt}{t} + 2CCvvdt = 2adt$

ita vt hic nulla potestas indefinita ipsius t occurrat. Si hic porro ponatur $v = \frac{\alpha}{t} + s$, fiet

$$-\frac{(n+2)C\alpha dt}{tt} + (n+2)Cds + \frac{nCsdt}{t} + 2CCssdt = 2adt$$
$$+\frac{nC\alpha dt}{tt} \qquad + \frac{4CC\alpha sdt}{t}$$
$$+\frac{2CC\alpha\alpha dt}{tt}$$

vbi si capiatur $\alpha = \frac{-n}{4C}$, orietur

$$(n+2)Cds + 2CCssdt = 2adt - \frac{n(n+4)dt}{8tt}$$

quae forma simplicissima videtur.

Theorema.

837. Sumto elemento dx constante, si aequationi differentio-differentiali

$$ddy + Pdxdy + Qydx^2 = 0$$

satisfaciant integralia particularia $y = M$ et $y = N$, ita vt ratio $M:N$ non sit constans, erit eius integrale completum $y = \alpha M + \beta N$.

Demon-

Demonstratio.

Cum valores $y=M$ et $y=N$ satisfaciant aequationi propositae erit

$ddM+PdxdM+QMdx^2=0$ et $ddN+PdxdN+QNdx^2=0$

vnde patet ponendo $y=\alpha M+\beta N$ aequationi quoque satisfieri, cum fiat:

$$\left.\begin{array}{l}+\alpha(ddM+PdxdM+QMdx^2)\\+\beta(ddN+PdxdN+QNdx^2)\end{array}\right\}=0.$$

Quoniam vero hoc integrale $y=\alpha M+\beta N$ duas constantes α et β complectitur, quas pro lubitu definire licet, id completum sit necesse est, nisi forte N sit multiplum ipsius M.

Coroll. 1.

838. Ex datis ergo duobus integralibus particularibus huiusmodi aequationis, eius integrale completum formari potest, siquidem illa duo integralia sint inter se diuersa.

Coroll. 2.

839. Cum posito $y=e^{\int udx}$ seu $u=\frac{dy}{ydx}$ prodeat $du+uudx+Pudx+Qdx=0$. Si huic aequationi satisfaciant valores $u=\frac{dM}{Mdx}$ et $u=\frac{dN}{Ndx}$, eidemquoque satisfaciet valor $u=\frac{\alpha dM+\beta dN}{(\alpha N+\beta M)dx}$.

Coroll. 3.

840. Si ergo aequationis $du+uudx+Pudx+Qdx=0$ habeantur duo integralia particularia $u=R$ et $u=S$ ob $M=e^{\int R dx}$ et $N=e^{\int S dx}$ integrale completum erit

$$u=\frac{\alpha e^{\int R dx}R+\beta e^{\int S dx}S}{\alpha e^{\int R dx}+\beta e^{\int S dx}} \text{ siue}$$

$$u=R+\frac{\beta e^{\int S dx}(S-R)}{\alpha e^{\int R dx}+\beta e^{\int S dx}}.$$

Scholion.

841. Maximi momenti est haec obseruatio, quod in huiusmodi aequationibus ex cognitis binis integralibus particularibus integrale completum assignari possit. Plerumque autem cognito vno integrali particulari, in eo signum radicale inesse solet, ob cuius ambiguitatem duo simul integralia particularia innotescunt. Ita si aequationi

$$du+uudx+Pudx+Qdx=0$$

satisfaciat valor $u=T+\sqrt{V}$, eidem satisfaciet $u=T-\sqrt{V}$, vnde integrale completum erit:

$$u=T+\sqrt{V}-\frac{2\beta\sqrt{V}}{\alpha e^{2\int dx\sqrt{V}}+\beta} \text{ seu}$$

$$u=T+\frac{\alpha e^{2\int dx\sqrt{V}}\sqrt{V}-\beta\sqrt{V}}{\alpha e^{2\int dx\sqrt{V}}+\beta}.$$

Ac

Ac si forte sit $\sqrt{V}$ imaginarium puta $\sqrt{V} = X\sqrt{-1}$ ob $e^{\pm\int dx\sqrt{V}} = \operatorname{cos.}\int X dx \pm \sqrt{-1}.\operatorname{sin.}\int X dx$ erit

$$u = T + \frac{(\alpha-\beta)\operatorname{cos.}\int X dx + (\alpha+\beta)\operatorname{sin.}\int X dx.\sqrt{-1}}{(\alpha+\beta)\operatorname{cos.}\int X dx + (\alpha-\beta)\operatorname{sin.}\int X dx.\sqrt{-1}} X\sqrt{-1}$$

seu posito $(\alpha-\beta)\sqrt{-1} = \gamma$ et $\alpha+\beta = \delta$

$$u = T + \frac{\gamma\operatorname{cos.}\int X dx - \delta\operatorname{sin.}\int X dx}{\delta\operatorname{cos.}\int X dx + \gamma\operatorname{sin.}\int X dx}.X \text{ vel etiam}$$

$$u = T + X\operatorname{tang.}(\int X dx + \zeta).$$

Problema 102.

842. Sumto elemento dx constante, inuenire integrale completum huius aequationis differentio-differentialis:

$$ddy + Adydx + Bydx^2 = 0.$$

Solutio.

Posito $y = e^{\int u dx}$ prodit haec aequatio

$$du + uudx + Audx + Bdx = 0 \text{ siue}$$

$$dx = \frac{-du}{uu + Au + B}$$

cui satisfit tribuendo ipsi u eiusmodi valorem constantem, vt euadat $uu + Au + B = 0$, qui sunt

$$u = -\tfrac{1}{2}A \pm \sqrt{(\tfrac{1}{4}AA - B)}.$$

Hinc ergo cum habeantur bina integralia particularia $y = e^{\int u dx}$, posito $\sqrt{(\tfrac{1}{4}AA - B)} = n$ erit integrale completum

$$y = e^{-\frac{1}{2}Ax}(\alpha e^{nx} + \beta e^{-nx})$$

ac

ac si sit n numerus imaginarius puta $n = m\sqrt{-1}$ erit

$$y = e^{-\frac{1}{2}Ax}(\alpha \operatorname{cof.} mx + \beta \operatorname{fin.} mx) = Ce^{-\frac{1}{2}A} \operatorname{fin.}(mx + \gamma).$$

Sin autem sit $n = 0$ prodibit

$$y = e^{-\frac{1}{2}Ax}(\alpha + \beta x).$$

Coroll. 1.

843. Ad integrale ergo aequationis propositae inueniendum, resolui oportet aequationem algebraicam $uu + Au + B = 0$ quae oritur ex proposita $ddy + Adydx + Bydx^2 = 0$, si loco y, dy, ddy scribatur u^0, u^1, u^2 et elementum dx reiiciatur, tum enim binae radices illius aequationis dabunt integrale completum.

Coroll. 2.

844. Scilicet si aequationis $uu + Au + B = 0$ factores sint $(u + f)(u + g)$, ob valores $u = -f$ et $u = -g$ integrale completum erit $y = \alpha e^{-fx} + \beta e^{-gx}$. At si sit $g = f$ erit $y = e^{-fx}(\alpha + \beta x)$.

Coroll. 3.

845. Si aequatio $uu + Au + B = 0$ habeat factores imaginarios, quo casu huiusmodi formam habebit

$uu + 2fu \operatorname{cof.} \zeta + ff = 0$ erit $u = -f \operatorname{cof.} \zeta \pm f\sqrt{-1} . \operatorname{fin.} \zeta$

hincque

hincque integrale completum erit:

$$y = e^{-fx\cos.\zeta}(\alpha\cos.fx\sin.\zeta + \beta\sin.fx\sin.\zeta) \text{ siue}$$

$$y = Ce^{-fx\cos.\zeta}\sin.(fx\sin.\zeta + \gamma).$$

Scholion.

846. Idem integrale completum reperitur methodo consueta ex aequatione $dx = \frac{-du}{uu+Au+B}$, posito enim $uu+Au+B=(u+f)(u+g)$ erit

$$(g-f)dx = \frac{du}{u+g} - \frac{du}{u+f} \text{ et } Ce^{(g-f)x} = \frac{u+g}{u+f}$$

vnde fit

$$u = \frac{g - Cfe^{(g-f)x}}{Ce^{(g-f)x} - 1} \text{ seu}$$

$$u = \frac{-\alpha fe^{gx} + \beta ge^{fx}}{\alpha e^{gx} - \beta e^{fx}}.$$

Tum vero fit

$$\int u\,dx = \int \frac{\alpha fe^{(g-f)x} - \beta g}{\alpha e^{(g-f)x} - \beta}dx = -\int \frac{u\,du}{(u+f)(u+g)}$$

ideoque

$$\int u\,dx = \frac{f}{g-f}l(u+f) - \frac{g}{g-f}l(u+g).$$

Hinc

$$y = e^{\int u\,dx} = C(u+f)^{\frac{f}{g-f}}(u+g)^{\frac{-g}{g-f}}.$$

At est

$$u+f = \frac{\beta(g-f)e^{fx}}{\alpha e^{gx} - \beta e^{fx}} \text{ et } u+g = \frac{\alpha(g-f)e^{gx}}{\alpha e^{gx} - \beta e^{fx}}$$

vnde colligitur mutando constantem C

$$y = \frac{C e^{\frac{ffx}{g-f}} e^{\frac{-ggx}{g-f}}}{(\alpha e^{gx} - \beta e^{fx})^{\frac{f-g}{g-f}}} = C e^{-(f+g)x}(\alpha e^{gx} - \beta e^{fx})$$

seu $y = \alpha e^{-fx} + \beta e^{-gx}$ vt ante. Hinc ergo patet, quantum subsidium afferat formatio integralis completi ex binis particularibus.

Problema 103.

847. Sumto elemento dx constante, si proponatur haec aequatio differentio-differentialis

$$ddy + \frac{A dy dx}{x} + \frac{B y dx^2}{xx} = 0$$

eius integrale completum inuenire.

Solutio.

Ponatur $dy = pdx$ et $dp = qdx$, vt habeamus:

$$q + \frac{Ap}{x} + \frac{By}{xx} = 0 \text{ seu } q = \frac{-Ap}{x} - \frac{By}{xx}.$$

Sit nunc $p = \frac{uy}{x}$, erit $dy = \frac{uydx}{x}$ et

$$dp = \frac{udy + ydu}{x} - \frac{uydx}{xx} = -\frac{Ady}{x} - \frac{Bydx}{xx}$$

vnde colligimus

$$\frac{dy}{y} = \frac{udx}{x} = \frac{udx - Bdx - xdu}{xu + Ax} \text{ seu}$$

$$xdu + Bdx + uudx + (A-1)udx = 0$$

hincque

hincque $\frac{dx}{x} = \frac{-du}{uu+(A-1)u+B}$; cui particulariter satisfit ponendo $uu+(A-1)u+B=0$.

Sit primo $uu+(A-1)u+B=(u+f)(u+g)$, erit particulariter $ly=-flx$ et $y=x^{-f}$, similique modo $y=x^{-g}$, vnde integrale completum erit

$$y=\alpha x^{-f}+\beta x^{-g}.$$

Si sit $g=f$ statuatur $g=f-\omega$ euanescente ω erit

$$x^{-g}=x^{-f}.x^{\omega}=x^{-f}(1+\omega lx),$$

ergo hoc casu fit

$$y=x^{-f}(\alpha+\beta lx)$$

sit denique

$$uu+(A-1)u+B=uu+2fu\cos.\zeta+ff \text{ erit}$$

$$u=-f(\cos\zeta\pm\sqrt{-1}.\sin.\zeta),$$

ergo particulariter

$$y=x^{-f\cos.\zeta}.x^{\pm f\sqrt{-1}.\sin.\zeta}=x^{-f\cos.\zeta}(\cos.(f\sin.\zeta\, lx)\pm\sqrt{-1}.\sin.(f\sin.\zeta\, lx))$$

quare integrale completum erit

$$y=Cx^{-f\cos.\zeta}\sin.(f\sin.\zeta.lx+\gamma).$$

Coroll. 1.

848. Huius ergo aequationis:

$$ddy+(f+g+1)\frac{dydx}{x}+\frac{fgydx^2}{xx}=0$$

integrale completum est

$$y=\alpha x^{-f}+\beta x^{-g}$$

Huius autem

$$ddy + (2f+1)\frac{dydx}{x} + \frac{ffydx^2}{xx} = 0$$

integrale completum est

$$y = x^{-f}(\alpha + \beta\, lx).$$

Coroll. 2.

849. At si aequatio proposita huiusmodi formam habuerit

$$ddy + (1 + 2f\cos.\zeta)\frac{dydx}{x} + \frac{ffydx}{xx} = 0$$

tum eius integrale completum erit:

$$y = Cx^{-f\cos.\zeta}\sin.(f\sin.\zeta.lx + \gamma).$$

Scholion.

850. Similem resolutionem quoque admittit haec aequatio differentio-differentialis:

$$ddy - \frac{ndydx}{x} + Ax^n dydx + Bx^{2n}ydx^2 = 0.$$

Ponatur enim $dy = x^n yudx$, et cum sit:

$$ddy = x^n ydxdu + nx^{n-1}yudx^2 + x^{2n}yu^2dx^2$$

erit per y diuidendo

$$x^n dxdu + nx^{n-1}udx^2 + x^{2n}uudx^2 - nx^{n-1}udx^2 + Ax^{2n}udx^2 + Bx^{2n}dx^2 = 0$$

hinc $du + x^n uudx + Ax^n udx + Bx^n dx = 0$

ideoque $x^n dx = \frac{-du}{uu + Au + B}$

cui

cui particulariter ſatisfit ponendo $uu+Au+B=0$ vnde u duplicem conſequitur valorem conſtantem quorum alter ſit $u=-f$ alter $u=-g$. Quocirca integralia particularia erunt

$$y=e^{\frac{-fx^{n+1}}{n+1}} \quad \text{et} \quad y=e^{\frac{-gx^{n+1}}{n+1}}$$

ſit breuitatis gratia $\frac{x^{n+1}}{n+1}=t$, erit integrale completum

$$y=\alpha e^{-ft}+\beta e^{-gt}$$

pro caſu ſcilicet

$$uu+Au+B=(u+f)(u+g).$$

At pro caſu

$$uu+Au+B=(u+f)^2 \text{ erit } y=e^{-ft}(\alpha+\beta t).$$

Caſu autem quo

$$uu+Au+B=uu+2fu\operatorname{cof.}\zeta+ff \text{ erit}$$

$$y=Ce^{-ft\operatorname{cof.}\zeta}\operatorname{fin.}(ft\operatorname{fin.}\zeta+\gamma).$$

Haec integratio adeo ad hanc formam extendi poteſt ſumendo pro X functionem quamcunque ipſius x

$$ddy-\frac{dXdy}{X}+AXdydx+BXXydx^2=0.$$

Poſito enim $dy=Xuydx$ ſeu $\frac{dy}{y}=Xudx$ fit

$$Xdx=\frac{-du}{uu+Au+B}$$

vnde posito $\int X dx = t$ integrale completum se habebit vt ante. Scilicet

1) si $A = f + g$ et $B = fg$ erit integrale

$$y = \alpha e^{-ft} + \beta e^{-gt}$$

2) si $A = 2f$ et $B = ff$ erit integrale

$$y = e^{-ft}(\alpha + \beta t)$$

3) si $A = 2f \operatorname{cos.} \zeta$ et $B = ff$ erit integrale

$$y = C e^{-ft \operatorname{cos.} \zeta} \operatorname{sin.} (ft \sin \zeta + \gamma).$$

Problema 104.

851. Sumto elemento dx constante, si P, Q et X denotent functiones quascunque ipsius x, integrationem huius aequationis differentio-differentialis

$$ddy + P dy dx + Q y dx^2 = X dx^2$$

ad aequationem differentialem primi ordinis reducere.

Solutio.

Hic singulari modo procedamus, loco y binas nouas incognitas introducendo. Statuatur scilicet $y = uv$, et cum sit

$$dy = u dv + v du \text{ et } ddy = u ddv + 2 du dv + v ddu$$

aequatio nostra induet hanc formam

$$u ddv + 2 du dv + v ddu + P u dx dv + P v dx du + Q u v dx^2 = X dx^2.$$

Iam

Iam altera v ita determinetur, vt termini ipsa littera u affecti destruantur, quod fit si

$$ddv + Pdxdu + Qvdx^2 = 0$$

vnde per superiora v per x determinetur, quo facto superest haec aequatio:

$$2dudv + vddu + Pvdxdu = Xdx^2$$

vnde cum v iam detur per x quantitas u definiri debet. Ponatur $du = sdx$, eritque

$$vds + 2sdv + Psvdx = Xdx$$

quae multiplicata per $ve^{\int Pdx}$ integrabilis redditur; prodit enim

$$vvse^{\int Pdx} = \int e^{\int Pdx} Xvdx \text{ ideoque}$$

$$s = \frac{e^{-\int Pdx}}{vv} \int e^{\int Pdx} Xvdx \text{ et}$$

$$u = \int \frac{e^{-\int Pdx}dx}{vv} \int e^{\int Pdx} Xvdx.$$

Quare cum incognita v fuerit determinata ex aequatione

$$ddv + Pdxdv + Qvdx^2 = 0$$

integrale aequationis propositae erit

$$y = v\int \frac{e^{-\int Pdx}dx}{vv} \int e^{\int Pdx} Xvdx.$$

Coroll. 1.

Coroll. 1.

852. Vt integratio ad aequationem differentialem primi ordinis reuocetur ponatur $v = e^{\int t\,dx}$, et quantitas t definietur per hanc aequationem

$$dt + tt\,dx + Pt\,dx + Q\,dx = 0$$

quo facto integrale quaesitum erit

$$y = e^{\int t\,dx} \int e^{-\int (P+2t)\,dx}\, dx \int e^{\int (P+t)\,dx}\, X\,dx.$$

Coroll. 2.

853. Cum sit $(P+t)dx = -\frac{dt}{t} - \frac{Q\,dx}{t}$ erit $e^{\int (P+t)\,dx} = \frac{1}{t} e^{-\int \frac{Q\,dx}{t}}$ hincque

$$y = e^{\int t\,dx} \int e^{\int \frac{Q\,dx}{t} - \int t\,dx}\, t\,dx \int e^{-\int \frac{Q\,dx}{t}}\, X \frac{dx}{t}$$

vbi duplex integratio ad integrale completum perducit.

Scholion 1.

854. Alio modo qui propius ad ante vsitatum accedat, eadem integratio institui potest. Ponatur scilicet pro aequatione proposita $dy = ty\,dx + v\,dx$, vbi v certam ipsius x functionem designet ex functione X determinandam. Cum igitur sit

$$ddy = y\,dt\,dx + t\,dx(ty\,dx + v\,dx) + dv\,dx$$

erit facta substitutione

$$\left.\begin{aligned} & y\,dt\,dx + tty\,dx^2 + Pty\,dx^2 + Qy\,dx^2 \\ & + tv\,dx^2 + dv\,dx + Pv\,dx^2 - X\,dx^2 \end{aligned}\right\} = 0$$

cuius

cuius aequationis vtraque pars, tam ea quae per y multiplicatur, quam altera ab y libera feorfim nihilo aequetur, vnde has duas aequationes nanciscimur:

$$dt + ttdx + Ptdx + Qdx = 0 \text{ et}$$
$$dv + tvdx + Pvdx = Xdx$$

ex quarum illa t per x vt ante definiri debet, tum vero erit ex ista

$$e^{\int (P+t)dx} v = \int e^{\int (P+t)dx} X dx.$$

Iam vero ex aequatione affumta $dy - tydx = vdx$ colligitur:

$$e^{-\int tdx} y = \int e^{-\int tdx} v dx$$

vbi fi loco v valor modo inuentus fubftituatur, praecedens integralis forma obtinetur.

Scholion 2.

855. Ex hac operatione fequi videtur, aequationis propofitae

$$ddy + Pdydx + Qydx^2 = Xdx^2$$

integrationem neceffario pendere ab integratione huius

$$ddv + Pdvdx + Qvdx^2 = 0$$

quandoquidem hac conceffa illa exhiberi poteft. Minime tamen hinc viciffim colligere licet, fi pofterioris refolutio vires noftras fuperet, etiam priorem nullo modo integrari poffe, quin potuis facile eft

 infini-

infinitos casus exhibere, quibus prior integrationem admittat, cum tamen posterior irresolubilis existat. Sit enim $P=0$ et $Q=\alpha x$, atque certum est aequationem posteriorem $ddv+\alpha xvdx^2=0$ nulla adhuc methodo resolui posse, cum posito $v=e^{\int tdx}$, abeat in $dt+ttdx+\alpha xdx=0$; neque tamen hinc sequitur aequationem priorem

$$ddy+\alpha xydx^2=Xdx^2$$

semper esse intractabilem. Infiniti enim casus pro X assignari possunt, quibus integratio succedat. Sumta enim pro y functione quacunque ipsius x, reperietur pro X eiusmodi functio, vt aequationi valor pro y assumtus satisfaciat. Veluti posito $y=\frac{\beta x}{\alpha}$, ob $ddy=0$ fit $X=\beta xx$, atque aequationi

$$ddy+\alpha xydx^2=\beta xxdx^2$$

vtique satisfacit integrale $y=\frac{\beta x}{\alpha}$. Interim tamen hoc integrale tantum est particulare, ac dubium adhuc relinquitur, an etiam integrale completum exhiberi possit. At posito $y=\frac{\beta x}{x}+z$ pro integrali completo inueniendo prodit $ddz+\alpha xzdx^2=0$, quae cum resolutionem respuat, euidens est integrale completum etiam in genere exhiberi non posse, nisi simul altera aequatio integrationem admittat.

Proble-

Problema 105.

856. Sumto elemento dx constante, inuenire integrale completum huius aequationis differentio-differentialis

$$ddy + Adydx + Bydx^2 = Xdx^2$$

denotante X functionem quamcunque ipsius x.

Solutio.

Posito $y = uv$, aequatio proposita in duas sequentes resoluitur:

$$ddv + Advdx + Bvdx^2 = 0 \text{ et}$$

$$vddu + 2dvdu + Avdxdu = Xdx^2.$$

Quodsi ergo ex priore valor ipsius v per x definiatur, integrale completum ex posteriore ita se habebit, vt ob P = A sit:

$$y = v\int \frac{e^{-Ax}dx}{vv} \int e^{Ax} Xvdx,$$

vbi cum duplex integratio integrale completum producat, sufficiet pro v integrale particulare prioris aequationis assumsisse, id quod etiam ex solutione generali patebit. Cum igitur pro resolutione prioris aequationis formanda sit haec aequatio quadratica $tt + At + B = 0$ pro eius indole tres casus euolui conueniet.

I. Si $tt+At+B=(t+f)(t+g)$, vt sit $A=f+g$ et $B=fg$, erit $v=\alpha e^{-fx}+\beta e^{-gx}$. Sumatur primo tantum integrale particulare $v=e^{-fx}$ et ob $A=f+g$, fiet

$$y=e^{-fx}\int e^{(f-g)x}dx\int e^{gx}Xdx \text{ fit}$$

$$e^{(f-g)x}dx=dR \text{ et } \int e^{gx}Xdx=S \text{ vt fiat}$$

$$y=e^{-fx}\int SdR=e^{-fx}(RS-\int RdS)$$

at est $R=\frac{1}{f-g}e^{(f-g)x}$ vnde colligitur

$$y=\frac{1}{f-g}e^{-gx}S-\frac{1}{f-g}e^{-fx}\int e^{fx}Xdx \text{ siue}$$

$$(f-g)y=e^{-gx}\int e^{gx}Xdx-e^{-fx}\int e^{fx}Xdx$$

quod idem integrale prodiisset, si altero particulari $v=e^{-gx}$ vsi essemus.

In genere autem sumto $v=\alpha e^{-fx}+\beta e^{-gx}$ statuatur vt ante,

$$\frac{e^{-(f+g)x}dx}{vv}=dR \text{ et } \int e^{(f+g)x}Xvdx=S$$

vt sit pariter

$$y=v\int SdR=v(RS-\int RdS).$$

Fingamus

$$R=\frac{Ce^{\lambda x}}{v} \text{ et ob } dv=-dx(\alpha fe^{-fx}+\beta ge^{-gx}) \text{ erit}$$

$$dR=\frac{Ce^{\lambda x}dx(\alpha\lambda e^{-fx}+\beta\lambda e^{-gx}+\alpha fe^{-fx}+\beta ge^{-gx})}{vv}$$

sit

fit iam $\lambda = -g$ et $C\alpha(f-g) = 1$, vt dR datum adipifcatur valorem, ob $C = \frac{1}{\alpha(f-g)}$ erit

$$R = \frac{e^{-gx}}{\alpha(f-g)v} \text{ et } RdS = \frac{1}{\alpha(f-g)} e^{fx} X dx$$

tum vero

$$S = \alpha \int e^{gx} X dx + \beta \int e^{fx} X dx$$

vnde conficitur:

$$y = v\left(\frac{e^{-gx}}{(f-g)v} \int e^{gx} X dx + \frac{\beta e^{-gx}}{\alpha(f-g)v} \int e^{fx} X dx - \frac{1}{\alpha(f-g)} \int e^{fx} X dx\right)$$

feu prorfus vt ante

$$(f-g)y = e^{-gx} \int e^{gx} X dx - e^{-fx} \int e^{fx} X dx.$$

II. Si $tt + At + B = (t+f)^2$ feu $A = 2f$ et $B = ff$, erit ex priori aequatione $v = e^{-fx}(\alpha + \beta x)$. Ponatur vt ante

$$\frac{e^{-2fx} dx}{vv} = dR \text{ et } \int e^{2fx} X v dx = S$$

vt habeatur $y = v(RS - \int R dS)$. Cum ergo fit

$$dR = \frac{dx}{(\alpha + \beta x)^2} \text{ erit } R = -\frac{1}{\beta(\alpha + \beta x)} = -\frac{e^{-fx}}{\beta v} \text{ et}$$

$$S = \alpha \int e^{fx} X dx + \beta \int e^{fx} X x dx = \int e^{fx} X dx (\alpha + \beta x)$$

quare $vRS = -\frac{\alpha}{\beta} e^{-fx} \int e^{fx} X dx - e^{-fx} \int e^{fx} X x dx$ et

$$\int R dS = -\frac{1}{\beta} \int e^{fx} X dx,$$

vnde conficitur

$$y = e^{-fx} x \int e^{fx} X\,dx - e^{-fx} \int e^{fx} X x\,dx$$

ſeu cum ſit

$$d.e^{fx} y = dx \int e^{fx} X\,dx$$

erit ſuccinctius

$$y = e^{-fx} \int dx \int e^{fx} X\,dx.$$

III. Si $tt + tA + B = tt + 2ft\,\mathrm{cof.}\,\zeta + ff$ ſeu $A = 2f\,\mathrm{cof.}\,\zeta$ et $B = ff$ erit

$$v = e^{-fx\,\mathrm{cof.}\,\zeta}\,\mathrm{fin}\,(fx\,\mathrm{fin.}\,\zeta + \gamma).$$

Ponatur

$$\frac{e^{-2fx\,\mathrm{cof.}\,\zeta}\,dx}{vv} = \frac{dx}{\mathrm{fin.}(fx\,\mathrm{fin.}\,\zeta + \gamma)^2} = dR \text{ et}$$

$$e^{2fx\,\mathrm{cof.}\,\zeta} X v\,dx = e^{fx\,\mathrm{cof.}\,\zeta} X\,dx\,\mathrm{fin.}(fx\,\mathrm{fin.}\,\zeta + \gamma) = dS$$

vt obtineatur $y = vRS - v\int R\,dS$. At eſt

$$R = -\frac{1}{f\,\mathrm{fin.}\,\zeta} \cdot \frac{\mathrm{cof.}(fx\,\mathrm{fin.}\,\zeta + \gamma)}{\mathrm{fin.}(fx\,\mathrm{fin.}\,\zeta + \gamma)}; \text{ hincque}$$

$$vRS = -\frac{1}{f\,\mathrm{fin.}\,\zeta} e^{-fx\,\mathrm{cof.}\,\zeta}\,\mathrm{cof.}(fx\,\mathrm{fin.}\,\zeta + \gamma) \int e^{fx\,\mathrm{cof.}\,\zeta} X\,dx\,\mathrm{fin.}(fx\,\mathrm{fin}\,\zeta + \gamma)$$

$$\text{et } \int R\,dS = -\frac{1}{f\,\mathrm{fin.}\,\zeta} \int e^{fx\,\mathrm{cof.}\,\zeta} X\,dx\,\mathrm{cof.}(fx\,\mathrm{fin.}\,\zeta + \gamma).$$

Quocirca obtinebitur

$$fy\,\mathrm{fin.}\,\zeta = + e^{-fx\,\mathrm{cof.}\,\zeta}\,\mathrm{fin.}(fx\,\mathrm{fin.}\,\zeta + \gamma) \int e^{fx\,\mathrm{cof.}\,\zeta} X\,dx\,\mathrm{cof.}(fx\,\mathrm{fin.}\,\zeta + \gamma)$$
$$- e^{-fx\,\mathrm{cof.}\,\zeta}\,\mathrm{cof.}(fx\,\mathrm{fin.}\,\zeta + \gamma) \int e^{fx\,\mathrm{cof.}\,\zeta} X\,dx\,\mathrm{fin}\,(fx\,\mathrm{fin}\,\zeta + \gamma).$$

Coroll. 1.

Coroll. 1.

857. In hoc postremo integrali si ponamus $fx\,\text{sin.}\,\zeta = \Phi$ erit

$$\int e^{fx\cos.\zeta} y\,\text{sin.}\,\zeta = (\text{sin.}\,\gamma\cos.\Phi + \cos.\gamma\,\text{sin.}\,\Phi)\int e^{fx\cos.\zeta} X dx(\cos.\gamma\cos.\Phi - \text{sin.}\,\gamma\,\text{sin.}\,\Phi)$$
$$+(\text{sin.}\,\gamma\,\text{sin.}\,\Phi - \cos.\gamma\cos.\Phi)\int e^{fx\cos.\zeta} X dx(\text{sin.}\,\gamma\cos.\Phi + \cos.\gamma\,\text{sin.}\,\Phi)$$

seu

$$\int e^{fx\cos.\zeta} y\,\text{sin.}\,\zeta = +\text{sin.}\,\gamma\cos.\gamma\cos.\Phi\int e^{fx\cos.\zeta} X dx\cos\Phi - \text{sin}\,\gamma^2\cos\Phi\int e^{fx\cos.\zeta} X dx\,\text{sin}\,\Phi$$
$$+\cos.\gamma^2\,\text{sin.}\,\Phi\int e^{fx\cos.\zeta} X dx\cos\Phi - \text{sin.}\,\gamma\cos\gamma\,\text{sin}\,\Phi\int e^{fx\cos.\zeta} X dx\,\text{sin}\,\Phi$$
$$+\text{sin.}\,\gamma^2\,\text{sin.}\,\Phi\int e^{fx\cos.\zeta} X dx\cos\Phi + \text{sin}\,\gamma\cos\gamma\,\text{sin}\,\Phi\int e^{fx\cos.\zeta} X dx\,\text{sin}\,\Phi$$
$$-\text{sin.}\,\gamma\cos.\gamma\cos.\Phi\int e^{fx\cos.\zeta} X dx\cos\Phi - \cos\gamma^2\cos\Phi\int e^{fx\cos.\zeta} X dx\,\text{sin}\,\Phi$$

vnde patet angulum γ prorsus ex calculo excedere; fit enim

$$\int e^{fx\cos.\zeta} y\,\text{sin.}\,\zeta = \text{sin.}\,\Phi\int e^{fx\cos.\zeta} X dx\cos.\Phi - \cos.\Phi\int e^{fx\cos.\zeta} X dx\,\text{sin.}\,\Phi.$$

Coroll. 2.

858. Cum igitur loco vnius aequationis duas formauerimus integrandas, vidimns sufficere, si alterius integrale saltem particulare fuerit cognitum. In ambobus enim praecedentibus casibus constantes α et β integrale completum praebentes ex calculo sponte euanuerunt, et casu tertio constans γ itidem excessit.

Exemplum.

859. *Sumto elemento* dx *constante*, *inuenire integrale huius aequationis*:

$$ddy + Adydx + Bydx^2 = dx^2(n(n-1)x^{n-2} + nAx^{n-1} + Bx^n).$$

Hoc

Hoc exemplum ita est comparatum, vt ei manifesto satisfaciat valor $y=x^n$, qui eius integrale particulare constituit. Ad completum ergo inueniendum, sit $A=f+g$ et $B=fg$ et cum sit

$$X=fgx^n+n(f+g)x^{n-1}+n(n-1)x^{n-2} \text{ erit}$$
$$\int e^{gx}X\,dx=fe^{gx}x^n+ne^{gx}x^{n-1}+\alpha \text{ et}$$
$$\int e^{fx}X\,dx=ge^{fx}x^n+ne^{fx}x^{n-1}+\beta$$

vnde ex forma inuenta prodit integrale completum:

$$(f-g)y=fx^n+nx^{n-1}+\alpha e^{-gx}-gx^n-nx^{n-1}-\beta e^{-fx}$$

seu $y=x^n+\frac{\alpha}{f-g}e^{-gx}-\frac{\beta}{f-g}e^{-fx}$

vel mutata constantium forma:

$$y=x^n+\alpha e^{-fx}+\beta e^{-gx}$$

si sit $g=f$, ponatur $g=f+\omega$, existente $\omega=0$, et ob $e^{-gx}=e^{-fx}.e^{-\omega x}=e^{-fx}(1-\omega x)$, vnde pro $\alpha+\beta$ et $-\beta\omega$ scribendo α et β erit $y=x^n+e^{-fx}(\alpha+\beta x)$.

Sin autem sit

$$f=a+b\sqrt{-1} \text{ et } g=a-b\sqrt{-1},$$

cum fiat

$$y=x^n+e^{-ax}(\alpha e^{-bx\sqrt{-1}}+\beta e^{bx\sqrt{-1}}) \text{ ob}$$
$$e^{\pm bx\sqrt{-1}}=\text{cos}.\,bx\pm\sqrt{-1}.\,\text{sin}.\,bx,$$

mutata forma constantium habebimus:

$$y=x^n+e^{-ax}(\alpha\,\text{cos}.\,bx+\beta\,\text{sin}.\,bx)$$

Scholion.

Scholion.

860. In genere autem ſi huiusmodi aequationis

$$ddy + A dy dx + B y dx^2 = X dx^2$$

conſtet integrale particulare, ſeu valor ipſi ſatisfaciens $y = t$, iſitegrale completum facile reperitur ponendo $y = t + z$. Cum enim per hypotheſin ſit

$$ddt + A dt dx + B t dx^2 = X dx^2$$

facta hac ſubſtitutione orietur:

$$ddz + A dz dx + B z dx^2 = 0$$

vnde ſi $A = f + g$ et $B = fg$ colligitur

$$z = \alpha e^{-fx} + \beta e^{-gx}$$

ſicque integrale completum erit:

$$y = t + \alpha e^{-fx} + \beta e^{-gx}$$

quemadmodum etiam in exemplo allato inuenimus.

Problema 106.

861. Sumto elemento dx conſtante, ſi proponatur haec aequatio differentio-differentialis:

$$ddy - \frac{n dy dx}{x} + A x^n dy dx + B x^{2n} y dx^2 = X dx^2$$

exiſtente X functione quacunque ipſius x eius integrale completum inueſtigare.

Solutio.

Resolutio huius aequationis vt supra ex positione $y=uv$ deriuari posset, sed alia methodo hic vtentes leui substitutione eam ad formam Probl. praeced. reducamus. Scilicet ponamus $x^n dx=dt$, vt sit $x^{n+1}=(n+1)t$ qua substitutione functio X abeat in T functionem quandam ipsius t. Ne autem assumtio elementi dx constantis turbet, hanc conditionem tollamus ponendo $dy=pdx$ et $dp=qdx$, habebimusque

$$q-\frac{np}{x}+Ax^n p+Bx^{2n}y=X.$$

Cum nunc sit $dx=\frac{dt}{x^n}$ erit $p=\frac{x^n dy}{dt}$ hincque sumto iam elemento dt constante:

$$dp=\frac{nx^{n-1}dxdy}{dt}+\frac{x^n ddy}{dt}=qdx=\frac{qdt}{x^n} \text{ ergo}$$

$$q=\frac{nx^{2n-1}dy}{dt}+\frac{x^{2n}ddy}{dt^2}$$

sicque nostra aequatio erit

$$\frac{x^{2n}ddy}{dt^2}+\frac{nx^{n-1}dy}{dt}-\frac{nx^{n-1}dy}{dt}+\frac{Ax^{2n}dy}{dt}+Bx^{2n}y=X$$

sit $Xx^{-2n}=\Theta$, quae quantitas posito $x^{n+1}=(n+1)t$ vt functio ipsius t spectari potest, sicque fiet:

$$ddy+Adydt+Bydt^2=\Theta dt^2$$

in

in qua aequatione elementum dt constans est assumtum, cuius ergo integrale per superiora datur.

I. Si $A = f + g$ et $B = fg$, erit integrale

$$(f-g)y = e^{-gt}\int e^{gt}\Theta\,dt - e^{-ft}\int e^{ft}\Theta\,dt$$

vbi restitutis valoribus $dt = x^n dx$ et $\Theta = Xx^{-2n}$, retento breuitatis gratia $t = \frac{1}{n+1}x^{n+1}$, valor ipsius y ita per x exprimetur:

$$(f-g)y = e^{-gt}\int e^{gt}x^{-n}X\,dx - e^{-ft}\int e^{ft}x^{-n}X\,dx.$$

II. Si $A = 2f$ et $B = ff$ erit integrale

$$y = e^{-ft}t\int e^{ft}\Theta\,dt - e^{-ft}\int e^{ft}\Theta\,t\,dt \text{ seu}$$

$$y = e^{-ft}\int dt\int e^{ft}\Theta\,dt$$

quod ergo per x ita exprimetur:

$$y = e^{-ft}\int x^n dx\int e^{ft}x^{-n}X\,dx.$$

III. Si denique $A = 2f\,\text{cos.}\,\zeta$ et $B = ff$ erit integrale

$$fe^{ft\,\text{cos.}\,\zeta}y\,\text{sin.}\,\zeta = \text{sin.}\,\Phi\int e^{ft\,\text{cos.}\,\zeta}\Theta\,dt\,\text{cos.}\,\Phi - \text{cos.}\,\Phi\int e^{ft\,\text{cos.}\,\zeta}\Theta\,dt\,\text{sin.}\,\Phi$$

existente $\Phi = ft\,\text{sin.}\,\zeta$ seu $\Phi = \frac{f\,\text{sin.}\,\zeta}{n+1}x^{n+1}$, ob $t = \frac{1}{n+1}x^{n+1}$.

Quare aequationis propositae integrale erit:

$$fe^{ft\,\text{cos.}\,\zeta}y\,\text{sin.}\,\zeta = \text{sin.}\,\Phi\int e^{ft\,\text{cos.}\,\zeta}x^{-n}X\,dx\,\text{cos.}\,\Phi - \text{cos.}\,\Phi\int e^{ft\,\text{cos.}\,\zeta}x^{-n}X\,dx\,\text{sin.}\,\Phi.$$

Coroll. 1.

862. Si $n=0$, aequatio propofita abit in eam ipfam, quam problemate praecedente tractauimus, fitque $t=x$, vnde etiam integrale eodem redit.

Coroll. 2.

863. Sin autem fit $n=-1$, aequatio noftra fit

$$ddy+(A+1)\frac{dy\,dx}{x}+\frac{By\,dx^2}{xx}=X\,dx^2$$

vbi ergo erit $t=lx$ et $e^{\lambda t}=x^{\lambda}$, tum vero pro cafu tertio angulus $\Phi=f\,\text{fin.}\,\zeta.\,lx$.

Scholion.

864. Methodus qua hic vfi fumus, huiusmodi aequationes differentio-differentiales integrandi, haud fatis naturalis videtur, cum ad has quafi folas formas fit adftricta. Quoniam igitur in aequationibus differentialibus primi gradus inuentio factorum, quibus eae per fe integrabiles reddantur, infignem fructum polliceri videbatur, eius quoque vfum in aequationibus differentialibus fecundi gradus oftendere conemur. Hic quidem nihil tam abfolutum expectare licet, quod ad omnes omnino aequationum formas pateat, fed quantillum etiam praeftare potuerimus, id haud contemnendum Analyfeos incrementum fpectari debebit. Hac autem methodo

cas

eas potiſſimum aequationes differentio-differentiales, in quibus altera variabilis y cum ſuis differentialibus vnam dimenſionem nusquam transgreditur, ſatis commode tractare licet, hincque via perſpicietur, quomodo eam magis excoli oporteat.

CAPVT V.

DE INTEGRATIONE AEQVATIONVM DIFFERENTIALIVM SECVNDI GRADVS, IN QVIBVS ALTERA VARIABILIS VNAM DIMENSIONEM NON SVPERAT, PER FACTORES.

Problema 107.

865.

Sumto elemento dx constante, si proponatur haec aequatio

$$ddy + Adxdy + Bydx^2 = Xdx^2$$

vbi X denotat functionem quamcunque ipsius x, inuenire functionem ipsius x, per quam haec aequatio multipticata fiat integrabilis.

Solutio.

Ponatur $dy = pdx$, vt habeatur forma differentialis primi gradus:

$$dp + Apdx + Bydx = Xdx$$

quae

quae multiplicata per V functionem quandam ipsius x fiat integrabilis; scilicet

$$Vdp + AVpdx + BVydx = VXdx$$

vbi cum posterius membrum $VXdx$ sit integrabile, idem in priori eueniat necesse est. At primo perspicuum est eius integralis partem fore Vp, vnde id ponatur $Vp+S$ vt sit $Vp+S=\int VXdx$, fietque

$$dS = -pdV + AVpdx + BVydx \text{ seu}$$

$$dS = dy\left(AV - \frac{dV}{dx}\right) + BVydx$$

quae forma integrabilis reddi potest sumendo $V=e^{\lambda x}$, erit enim

$$dS = e^{\lambda x}((A-\lambda)dy + Bydx) \text{ et } S = (A-\lambda)e^{\lambda x}y$$

vbi λ ita debet accipi, vt fiat

$$A\lambda - \lambda\lambda = B \text{ seu } \lambda\lambda - A\lambda + B = o.$$

Tum ergo erit

$$e^{\lambda x}p + (A-\lambda)e^{\lambda x}y = \int e^{\lambda x}Xdx \text{ seu}$$

$$dy + (A-\lambda)ydx = e^{-\lambda x}dx\int e^{\lambda x}Xdx$$

quae iam per $e^{(A-\lambda)x}$ multiplicata denuo fit integrabilis, datque:

$$e^{(A-\lambda)x}y = \int e^{(A-\lambda)x}dx\int e^{\lambda x}Xdx.$$

Cum

Cum λ fit vna radix aequationis $\lambda\lambda - A\lambda + B = o$, fi ambas eius radices ponamus f et g, vt fit $\lambda = f$ erit $A - \lambda = g$, et aequatio integralis:

$$e^{gx}y = \int e^{(g-f)x}dx\int e^{fx}Xdx \text{ feu}$$

$$e^{gx}y = \frac{1}{g-f}e^{(g-f)x}\int e^{fx}Xdx - \frac{1}{g-f}\int e^{gx}Xdx$$

quae abit in formam fupra inuentam

$$y = \frac{1}{g-f}e^{-fx}\int e^{fx}Xdx - \frac{1}{g-f}e^{-gx}\int e^{gx}Xdx.$$

Coroll. 1.

866. Aequatio ergo propofita feu inde nata

$$dp + Apdx + Bydx = Xdx$$

fit integrabilis fi ducatur in $e^{\lambda x}$ exiftente $\lambda\lambda - A\lambda + B = o$, ficque duplex habetur factor vel e^{fx} vel e^{gx}.

Coroll. 2.

867. Ea autem per factorem e^{fx} multiplicata, eius integrale erit

$$dy + gydx = e^{-fx}dx\int e^{fx}Xdx$$

ficque per integrationem ad aequationem differentialem primi gradus reducitur, quae denuo integrabilis redditur fi per e^{gx} multiplicetur.

Scholion.

868. Multiplicatorem V ita determinari oportebat, vt formula $dy(AV - \frac{dV}{dx}) + BVydx$ fieret per fe

se integrabilis. Tum autem cum V sit functio ipsius x, integrale erit $y(AV - \frac{dV}{dx})$, vnde fiat necesse est

$$AdV - \frac{ddV}{dx} = BVdx \text{ seu } ddV - Adx\,dV + BVdx^2 = 0$$

a cuius aequationis integratione pendet inuentio factoris quaesiti V. Sufficit autem eius integrale particulare sumsisse, dummodo enim aequatio proposita integrabilis reddatur, constans arbitraria pro integrali completo reddendo ipsa integratione introducitur.

Problema 108.

869. Sumto elemento dx constante, si proponatur haec aequatio

$$ddy + Pdydx + Qydx^2 = Xdx^2$$

vbi P, Q et X sint functiones quaecunque ipsius x, inuenire multiplicatorem V, qui sit functio ipsius x, quo illa aequatio integrabilis reddatur.

Solutio.

Quia aequatio per V multiplicata:

$$Vddy + VPdydx + VQydx^2 = VXdx^2$$

integrabilis existit, prioris partis integrale ponatur $Vdy + Sydx$, aliam enim formam habere nequit, ac fieri oportet:

$$VPdydx + VQydx^2 = dydV + Sdydx + ydSdx$$

vbi cum S fit necessario functio ipsius x, erit

$$VPdx = dV + Sdx \text{ et } VQdx = dS.$$

Inde autem est $S = VP - \frac{dV}{dx}$, quare multiplicator V definiri debet ex hac aequatione

$$VQdx = VdP + PdV - \frac{ddV}{dx} \text{ seu}$$

$$ddV - PdVdx + Vdx(Qdx - dP) = 0$$

quae ergo si resolui potuerit, vel si saltem eius integrale quodpiam particulare innotescat, vt habeatur multiplicator V, aequationis propositae integrale erit

$$Vdy + y(VPdx - dV) = dx\int VXdx$$

quae porro integrabilis redditur, si ducatur in $\frac{1}{VV}e^{\int Pdx}$, obtinebitur enim integrale

$$\frac{y}{V}e^{\int Pdx} = \int\frac{dx}{VV}e^{\int Pdx}\int VXdx \text{ seu}$$

$$y = e^{-\int Pdx}V\int e^{\int Pdx}\frac{dx}{VV}\int VXdx$$

quo duplici signo integrali gemina constans arbitraria introducitur, integrale completum constituens.

Coroll. 1.

870. Inuentio ergo multiplicatoris V pendet etiam a resolutione aequationis differentio-differentialis, quae autem proposita simplicior est aestimanda, quod functionem X non inuoluat, et quantitas V cum suis differentialibus dV et ddV vbique vnam dimensionem constituat.

Coroll. 2.

Coroll. 2.

871. Quodsi ergo ponatur $V = e^{\int v dx}$, quantitas v determinabitur per hanc aequationem differentialem primi gradus:

$$dv + vvdx - Pvdx + Qdx - dP = 0$$

cuius si saltem integrale particulare constet, integratio aequationis propositae absolui poterit.

Coroll. 3.

872. Dato autem multiplicatore V vicissim ratio aequationis propositae definitur, vt hoc modo integrabilis euadat. Erit enim vel

$$Q = \frac{dP}{dx} + \frac{PdV}{Vdx} - \frac{ddV}{Vdx^2} \text{ vel}$$

$$dP + \frac{PdV}{V} = Qdx + \frac{ddV}{Vdx} \text{ vel integrando}$$

$$PV = \frac{dV}{dx} + \int QVdx \text{ seu } P = \frac{dV}{Vdx} + \frac{\int QVdx}{V}.$$

Exemplum 1.

873. *Definire formam aequationis differentio-differentialis:*

$$ddy + Pdydx + Qydx^2 = Xdx^2$$

vt multiplicata per $e^{\lambda x}$ *integrabilis euadat.*

Posito multiplicatore $V = e^{\int v dx} = e^{\lambda x}$ erit $v = \lambda$, et satisfieri oportet huic aequationi

$$\lambda\lambda dx - \lambda Pdx + Qdx - dP = 0$$

vnde fit $Q=\lambda P-\lambda\lambda+\frac{dP}{dx}$. Primum ergo hoc euenit si fuerint P et Q constantes puta, $P=A$ et $Q=B$, ac tum λ definiri oportet ex hac aequatione $\lambda\lambda-A\lambda+B=0$, qui est casus supra tractatus. Praeterea vero qualiscunque functio P fuerit ipsius x, modo sit $Q=\lambda P-\lambda\lambda+\frac{dP}{dx}$, aequatio in $e^{\lambda x}$ ducta erit integrabilis, integrali existente:

$$e^{\lambda x}(dy+ydx(P-\lambda))=dx\int e^{\lambda x}Xdx \text{ seu}$$

$$dy+(P-\lambda)ydx=e^{-\lambda x}dx\int e^{\lambda x}Xdx$$

quae vlterius per $e^{\int Pdx-\lambda x}$ multiplicata et integrata dat

$$y=e^{-\int Pdx+\lambda x}\int e^{\int Pdx-2\lambda x}dx\int e^{\lambda x}Xdx.$$

Corollarium.

874. Sit $P=A+\alpha x$ et $Q=B+\beta x$, erit

$$B+\beta x=A\lambda+\alpha\lambda x-\lambda\lambda+\alpha \text{ ergo}$$

$$B=A\lambda-\lambda\lambda+\alpha \text{ et } \beta=\alpha\lambda,$$

vnde ob $\lambda=\frac{\beta}{\alpha}$, coefficientes A, B, α, β ita comparatos esse oportet, vt sit

$$B\alpha\alpha=A\alpha\beta-\beta\beta+\alpha^3 \text{ seu } B\alpha\alpha+\beta\beta=\alpha(A\beta+\alpha\alpha).$$

Exemplum 2.

875. *Definire formam aequationis differentio-differentialis:*

$$ddy+Pdydx+Qydx^2=Xdx^2$$

vt per $e^{\int vdx}$ *existente* $v=\frac{\lambda}{x}+\mu x^n$ *multiplicata fiat integrabilis.* Cum

Cum esse debeat

$$dv+vvdx-Pvdx+Qdx-dP=0\text{, erit}$$

$$\left.\begin{matrix}-\frac{\lambda}{xx}+\mu n x^{n-1}-\frac{\lambda P}{x}-\mu P x^{n}\\ +\frac{\lambda\lambda}{xx}+2\lambda\mu x^{n-1}+\mu\mu x^{2n}+Q-\frac{dP}{dx}\end{matrix}\right\}=0$$

ergo

$$Q=\frac{\lambda(1-\lambda)}{xx}-(2\lambda+n)\mu x^{n-1}-\mu\mu x^{2n}+\frac{\lambda P}{x}+\mu P x^{n}+\frac{dP}{dx}.$$

Ponamus $P=\frac{\alpha}{x}+\beta x^{n}$ erit

$$Q=\frac{1}{xx}(\lambda-\lambda\lambda+\alpha\lambda-\alpha)+x^{n-1}(\beta\lambda+\alpha\mu+\beta n-2\lambda\mu-n\mu)$$
$$+x^{2n}(\beta\mu-\mu\mu)$$

Sit $Q=\frac{\gamma}{xx}+\delta x^{n-1}+\varepsilon x^{2n}$, fierique oportet

$$\lambda\lambda-(\alpha+1)\lambda+\alpha+\gamma=0;\ \beta(\lambda+n)+\mu(\alpha-2\lambda-n)=\delta;$$
$$\text{et }\mu(\beta-\mu)=\varepsilon$$

vnde non solum pro multiplicatore litterae λ et μ, sed etiam certa relatio inter litteras α, β, γ, δ, ε definitur.

Veluti si sit $\gamma=0$, et $\delta=0$ erit $(\lambda-\alpha)(\lambda-1)=0$ vnde $\lambda=\alpha$; tum $(\beta-\mu)(\alpha+n)=0$ ergo $\alpha=\lambda=-n$ et $\mu\mu-\beta\mu+\varepsilon=0$. Scilicet aequatio:

$$ddy+dxdy(\beta x^{n}-\frac{n}{x})+\varepsilon x^{2n}ydx^{2}=Xdx^{2}$$

multiplicatorem recipit $e^{\int v dx}$ existente $v=-\frac{n}{x}+\mu x^{n}$ sumto μ ita vt sit $\mu\mu-\beta\mu+\varepsilon=0$. Erit ergo multiplicator

$$V=\frac{1}{x^{n}}e^{\frac{\mu}{n+1}x^{n+1}}\text{ et }e^{\int P dx}=\frac{1}{x^{n}}e^{\frac{\beta}{n+1}x^{n+1}}.$$

Quare si ponamus $\frac{1}{n+1}x^{n+1}=t$, erit

$$y=x^{n}e^{-\beta t}\frac{1}{x^{n}}e^{\mu t}\int e^{\beta t-2\mu t}x^{n}dx\int\frac{e^{\mu t}Xdx}{x^{n}}\quad \text{seu}$$

$$y=e^{(\mu-\beta)t}\int e^{(\beta-2\mu)t}dt\int\frac{e^{\mu t}Xdx}{x^{n}}.$$

Coroll. 1.

876. Si sumatur $\gamma=0$ et $\varepsilon=0$, erit $\mu=\beta$, $\beta(\alpha-\lambda)=\delta$ et $(\lambda-\alpha)(\lambda-1)=0$, hinc $\lambda=1$, et $\delta=(\alpha-1)\beta$, ideoque $P=\frac{\alpha}{x}+\beta x^{n}$; $Q=(\alpha-1)\beta x^{n-1}$, et aequationis

$$ddy+(\tfrac{\alpha}{x}+\beta x^{n})dxdy+(\alpha-1)\beta x^{n-1}y\,dx^{2}=Xdx^{2}$$

multiplicator $V=e^{\int v\,dx}$ existente $v=\frac{1}{x}+\beta x^{n}$ ita vt sit

$$V=xe^{\frac{\beta}{n+1}x^{n+1}}\quad \text{et}\quad e^{\int P\,dx}=x^{\alpha}e^{\frac{\beta}{n+1}x^{n+1}}.$$

Coroll. 2.

877. Hoc ergo casu posito $\frac{1}{n+1}x^{n+1}=t$ erit integrale:

$$y=x^{1-\alpha}\int x^{\alpha-2}e^{-\beta t}dx\int e^{\beta t}Xx\,dx$$

quae forma simplicius exhiberi nequit propterea quod in genere formula $e^{-\beta t}x^{\alpha-2}dx$ integrationem non admittit.

Scholion.

Scholion.

878. Cum igitur inuentio multiplicatorum, qui huiusmodi aequationem

$$ddy + Pdxdy + Qydx^2 = Xdx^2$$

integrabilem reddunt, resolutionem huius aequationis postulet

$$ddV - PdVdx + Vdx(Qdx - dP) = 0$$

quae in hac forma continetur

$$ddy + Pdxdy + Qydx^2 = 0$$

videndum est, quomodo hanc formam etiam per multiplicatores tractari oporteat. Cuius multiplicator si fingatur V functio quaedam ipsius x, iterum ad praecedentem formam

$$ddV - PdVdx + Vdx(Qdx - dP) = 0$$

deuenitur, atque si huius multiplicator statuatur $= U$ functioni ipsius x, hic definietur per hanc aequationem

$$ddU + PdUdx + QUdx = 0$$

ita vt sufficiat alteram harum duarum aequationum resoluisse. Ac supra quidem, vbi $y = uv$ posuimus, ad hanc posteriorem aequationem peruenimus: at mirum non est harum duarum aequationum alteram ab altera pendere, cum prior ex posteriori nascatur ponendo $U = e^{-\int Pdx} V$, posterior vero ex priori ponendo $V = e^{\int Pdx} U$, vti tentanti facile patebit.

Quoniam

Quoniam igitur hoc modo difficultatem, si quae occurrit, tollere non licet, inuestigandum est, an forte eiusmodi multiplicator, qui vtramque variabilem x et y cum suis differentialibus dx et dy seu $p=\frac{dy}{dx}$ inuoluat, negotium conficiat. At vero facile perspicitur exclusis differentialibus hoc fieri non posse; nam si multiplicator esset V functio ipsarum x et y, ex primo termino ddy nascetur integralis pars $V\,dy$, quae autem differentiata ponendo $dV=M\,dx+N\,dy$ inuolueret in differentiale partem $N\,dy^2$, in aequatione non occurrentem, quae etiam per reliquas integralis partes tolli non posset. Quare rem tentemus eiusmodi multiplicatoribus, qui etiam rationem differentialium $p=\frac{dy}{dx}$ complectantur; et cum ipsius y cum suis differentialibus vbique sit idem dimensionum numerus, eadem proprietas etiam in multiplicatore insit necesse est; si enim diuersae inessent, singulae seorsim negotium essent confecturae.

Problema 109.

879. Sumto elemento dx constante definire conditiones, vt multiplicator huius formae $Mp+Ny$, existente $p=\frac{dy}{dx}$ et M et N functionibus ipsius x integrabilem reddat hanc aequationem

$$ddy+P\,dx\,dy+Q\,y\,dx^2=0,$$

vbi P et Q sunt functiones ipsius x.

Solutio.

Solutio.

Ob $dy = pdx$, nostra aequatio est

$$dp + Ppdx + Qydx = 0$$

quae per $Mp + Ny$ multiplicata fit

$$Mpdp + Nydp + MPpdy + NPydy + NQyydx = 0$$
$$+ MQydy$$

quam integrabilem esse oportet. Ob terminos differentiali dp affectos pars integrabilis erit $\frac{1}{2}Mpp + Nyp$, vnde integrale ipsum statuatur $= \frac{1}{2}Mpp + Nyp + S$. cuius differentiale cum ipsam illam aequationem praebere debeat, habebimus:

$$\begin{aligned} dS = MPpdy &+ NPydy + NQyydx \\ &+ MQydy \\ -\tfrac{1}{2}ppdM &- ypdN \\ -Npdy & \end{aligned}$$

quam ergo formulam integrabilem esset oportet, quae cum tantum differentialia primi ordinis dx et dy complectatur, necesse est vt quantitas p ex calculo egrediatur. Posito ergo $dM = M'dx$ et $dN = N'dx$ ob $pdx = dy$, primus terminus continens p ad nihilum redigi debet vt fit

$$MPpdy - \tfrac{1}{2}M'pdy - Npdy = 0 \text{ seu}$$

$$MP - \tfrac{1}{2}M' - N = 0 \text{ vel } N = MP - \frac{dM}{2dx}.$$

Tum vero erit

$$dS = ydy(NP + MQ - N') + NQyydx$$

 cuius

cuius formulae integrale est

$$S = \tfrac{1}{2}yy(NP + MQ - N') \text{ vel } S = yy\int NQdx$$

quas duas formas congruere oportet, vnde fit

$$NP + MQ - \tfrac{dN}{dx} = 2\int NQdx \text{ seu}$$

$$NdP + PdN + MdQ + QdM - \tfrac{ddN}{dx} - 2NQdx = 0$$

quae aequatio cum illa $N = MP - \frac{dM}{2dx}$ iuncta conditiones quaesitas determinat, proditque tum aequatio integralis

$$\tfrac{1}{2}Mpp + Nyp + \tfrac{1}{2}yy(NP + MQ - \tfrac{dN}{dx}) = C.$$

Coroll. 1.

880. Si functiones P et Q dentur, indeque M et N definiri oporteat, ob $N = MP - \frac{dM}{2dx}$ erit $dN = MdP + PdM - \frac{ddM}{2dx}$ et functio M definietur per hanc aequationem:

$$\tfrac{d^3M}{2dx^2} - \tfrac{3PddM}{2dx} + (PP - \tfrac{3dP}{dx} + 2Q)dM + M(2PdP - \tfrac{ddP}{dx} - 2PQdx + dQ) = 0$$

quae ob differentialia tertii ordinis parum iuuat.

Coroll. 2.

881. Sin autem multiplicator $Mp + Ny$ detur, ipsa aequatio ita definitur, vt sit primo $P = \frac{N}{M} + \frac{dM}{2Mdx}$, vnde ex altera, quae est

$$dQ + \tfrac{QdM}{M} - \tfrac{2NQdx}{M} = \tfrac{ddN}{Mdx} - \tfrac{d.PN}{M}$$

haec-

haecque per $M e^{-2\int\frac{N dx}{M}}$ multiplicata integrale dat

$$MQe^{-2\int\frac{N dx}{M}} = \int e^{-2\int\frac{N dx}{M}}\left(\frac{ddN}{dx} - d.PN\right).$$

Coroll. 3.

882. Sit hoc integrale $= Z$ eritque

$$Z = e^{-2\int\frac{N dx}{M}}\left(\frac{dN}{dx} - PN\right) + \int e^{-2\int\frac{Ndx}{M}}\left(\frac{2NdN}{M} - \frac{2PNNdx}{M}\right)$$

quod posterius membrum pro P valore substituto abit in

$$\int e^{-2\int\frac{N dx}{M}}\left(\frac{2NdN}{M} - \frac{2N^3dx}{MM} - \frac{NNdM}{MM}\right)$$

cuius integrale est manifesto $e^{-2\int\frac{N dx}{M}}\frac{NN}{M}$, ita vt sit

$$Z = e^{-2\int\frac{N dx}{M}}\left(\frac{dN}{dx} - \frac{NdM}{2Mdx}\right) + C \text{ ideoque}$$

$$Q = \frac{C}{M}e^{2\int\frac{N dx}{M}} + \frac{dN}{Mdx} - \frac{NdM}{2MMdx}.$$

Coroll. 4.

883. Proposita ergo hac aequatione:

$$\frac{ddy}{dx} + \left(\frac{N}{M} + \frac{dM}{2Mdx}\right)dy + y\left(\frac{Cdx}{M}e^{2\int\frac{N dx}{M}} + \frac{dN}{M} - \frac{NdM}{2MM}\right) = 0$$

eam per $\frac{Mdy}{dx} + Ny$ multiplicando integrale fit

$$\frac{Mdy^2}{2dx^2} + \frac{Nydy}{dx} + \frac{1}{2}yy\left(Ce^{2\int\frac{N dx}{M}} + \frac{NN}{M}\right) = \text{Const.}$$

Scholion.

884. Cum ergo pro M et N quascunque functiones ipsius x accipere liceat, innumerabiles hinc nacti sumus aequationum differentio-differentialium formas, quas ope multiplicatoris $\frac{Mdy}{dx}+Ny$ integrare possumus. Forma scilicet generalis, quae hoc multiplicatore integrabilis redditur, est vt vidimus

$$\frac{ddy}{dx}+\frac{dy}{2Mdx}(dM+2Ndx)+\frac{y}{2MM}(2MdN-NdM+2CMe^{2\int\frac{Ndx}{M}}dx)$$

ipso integrali existente:

$$\frac{Mdy^2}{2dx^2}+\frac{Nydy}{dx}+\frac{1}{2}yy(\frac{NN}{M}+Ce^{2\int\frac{Ndx}{M}})$$

vbi perspicuum est partem exponentialem constanti C affectam vtrinque omitti posse, cum ea sola ista proprietate sit praedita. Quodsi partem exponentialem ad algebraicam reducamus ponendo $e^{2\int\frac{Ndx}{M}}=L$ erit $\frac{2Ndx}{M}=\frac{dL}{L}$ et $N=\frac{MdL}{2Ldx}$ hincque $dN=\frac{MddL}{2Ldx}+\frac{dLdM}{2Ldx}-\frac{MdL^2}{2LLdx}$; vnde ista forma

$$\frac{ddy}{dx}+\frac{dy}{2dx}(\frac{dM}{M}+\frac{dL}{L})+\frac{1}{2}y(\frac{ddL}{Ldx}+\frac{dLdM}{2LMdx}-\frac{dL^2}{LLdx}+\frac{2CLdx}{M})$$

quae per $\frac{Mdy}{dx}+\frac{MydL}{2Ldx}$ multiplicata integrale praebet:

$$\frac{Mdy^2}{2dx^2}+\frac{MydLdy}{2Ldx^2}+\frac{1}{2}yy(\frac{MdL^2}{4LLdx^2}+CL).$$

Vel si ponamus $\frac{dM}{M}+\frac{dL}{L}=\frac{2dK}{K}$ vt sit $M=\frac{KK}{L}$ erit nostra aequatio differentio-differentialis

$$\frac{ddy}{dx}+\frac{dy}{dx}\cdot\frac{dK}{K}+\frac{1}{2}y(d.\frac{dL}{Ldx}-\frac{dL^2}{2LLdx}+\frac{dKdL}{KLdx}+\frac{2CLLdx}{KK})=0$$

quae

quae per $\frac{KK}{L}\left(\frac{dy}{dx}+\frac{y\,dL}{2L\,dx}\right)$ multiplicata dabit integrale

$$\frac{KK}{2L}\left(\frac{dy^2}{dx^2}+\frac{y\,dL\,dy}{L\,dx^2}+yy\left(\frac{dL^2}{4LL\,dx^2}+\frac{CLL}{KK}\right)\right)=\text{Const.}$$

Exemplum 1.

885. Sit $K=x^m(a+x)^n$ et $L=x^\mu(a+x)^\nu$, erit $\frac{dK}{K\,dx}=\frac{m}{x}+\frac{n}{a+x}=\frac{ma+(m+n)x}{x(a+x)}$; $\frac{dL}{L\,dx}=\frac{\mu}{x}+\frac{\nu}{a+x}$ vnde coefficiens ipsius $\frac{1}{4}y\,dx$ erit

$$-\frac{\mu}{xx}-\frac{\nu}{(a+x)^2}-\frac{\mu\mu}{2xx}-\frac{\mu\nu}{x(a+x)}-\frac{\nu\nu}{2(a+x)^2}+\frac{m\mu}{xx}+\frac{m\nu+n\mu}{x(a+x)}+\frac{n\nu}{(a+x)^2}$$
$$+2Cx^{2\mu-2m}(a+x)^{2\nu-2n}$$

seu $$\frac{\mu(2m-\mu-2)}{2xx}+\frac{m\nu+n\mu-\mu\nu}{x(a+x)}+\frac{\nu(2n-\nu-2)}{2(a+x)^2}$$
$$+2Cx^{2\mu-2m}(a+x)^{2\nu-2n}$$

vbi sequentes casus notasse iuuabit.

I. Sit $m=\mu+1$ et $n=\nu$ erit ipsius $\frac{1}{4}y\,dx$ coefficiens:

$$\frac{\mu\mu+4C}{2xx}+\frac{\nu(\mu+1)}{x(a+x)}+\frac{\nu(\nu-2)}{2(a+x)^2}.$$

Hinc ista aequatio

$$\frac{ddy}{dx}+dy\left(\frac{\mu+1}{x}+\frac{\nu}{a+x}\right)+\frac{1}{4}y\,dx\left(\frac{\mu\mu+4C}{xx}+\frac{2\nu(\mu+1)}{x(a+x)}+\frac{\nu(\nu-2)}{(a+x)^2}\right)=0$$

multiplicata per $x^{\mu+2}(a+x)^\nu\left(\frac{dy}{dx}+\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)\frac{y}{2}\right)$ integrale dat:

$$\frac{1}{2}x^{\mu+2}(a+x)^\nu\left(\frac{dy^2}{dx^2}+\frac{y\,dy}{dx}\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)+\frac{1}{4}yy\left(\frac{\mu\mu+4C}{xx}+\frac{2\mu\nu}{x(a+x)}+\frac{\nu\nu}{(a+x)^2}\right)\right)=\text{Const.}$$

II. Sit $m=\mu+\frac{1}{2}$ et $n=\nu+\frac{1}{2}$ erit ipsius $\frac{1}{2}ydx$ coefficiens:

$$\frac{\mu(\mu-1)}{2xx}+\frac{2\mu\nu+\mu+\nu+4C}{2x(a+x)}+\frac{\nu(\nu-1)}{2(a+x)^2}.$$

Hinc ista aequatio:

$$\frac{ddy}{dx}+dy\left(\frac{2\mu+1}{2x}+\frac{2\nu+1}{2(a+x)}\right)+\frac{1}{4}ydx\left(\frac{\mu(\mu-1)}{xx}+\frac{2\mu\nu+\mu+\nu+4C}{x(a+x)}+\frac{\nu(\nu-1)}{(a+x)^2}\right)=0$$

multiplicata per $x^{\mu+1}(a+x)^{\nu+1}\left(\frac{dy}{dx}+\frac{1}{2}y\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)\right)$ integrale dabit:

$$\frac{1}{2}x^{\mu+1}(a+x)^{\nu+1}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)+\frac{1}{4}yy\left(\frac{\mu\mu}{xx}+\frac{2\mu\nu+4C}{x(a+x)}+\frac{\nu\nu}{(a+x)^2}\right)\right)=\text{Const.}$$

III. Sit $m=\mu$ et $n=\nu+1$ erit ipsius $\frac{1}{2}ydx$ coefficiens:

$$\frac{\mu(\mu-2)}{2xx}+\frac{\mu(\nu+1)}{x(a+x)}+\frac{\nu\nu+4C}{2(a+x)^2}.$$

Hinc ista aequatio:

$$\frac{ddy}{dx}+dy\left(\frac{\mu}{x}+\frac{\nu+1}{a+x}\right)+\frac{1}{4}ydx\left(\frac{\mu(\mu-2)}{xx}+\frac{2\mu(\nu+1)}{x(a+x)}+\frac{\nu\nu+4C}{(a+x)^2}\right)=0$$

multiplicata per $x^{\mu}(a+x)^{\nu+2}\left(\frac{dy}{dx}+\frac{1}{2}y\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)\right)$ dabit integrale:

$$\frac{1}{2}x^{\mu}(a+x)^{\nu+2}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)+\frac{1}{4}yy\left(\frac{\mu\mu}{xx}+\frac{2\mu\nu}{x(a+x)}+\frac{\nu\nu+4C}{(a+x)^2}\right)\right)=\text{Const.}$$

Coroll. 1.

886. Sit casu primo $\nu=2$ et $C=-\frac{1}{4}\mu\mu$ habebitur haec aequatio:

$$\frac{ddy}{dx}+\frac{(\mu+1)a+(\mu+3)x}{x(a+x)}dy+\frac{(\mu+1)ydx}{x(a+x)}=0$$

quae

quae per $x^{\mu+2}(a+x)^2\left(\frac{dy}{dx}+\frac{\mu a+(\mu+2)x}{2x(a+x)}y\right)$ multiplicata praebet integrale:

$$\tfrac{1}{2}x^{\mu+2}(a+x)^2\left(\frac{dy^2}{dx^2}+\frac{\mu a+(\mu+2)x}{x(a+x)}\cdot\frac{ydy}{dx}+yy\left(\frac{\mu}{x(a+x)}+\frac{1}{(a+x)^2}\right)\right)=\text{Const.}$$

Coroll. 2.

887. Sit casu tertio $\mu=2$ et $4C=-\nu\nu$ habebitur ista aequatio:

$$\frac{ddy}{dx}+dy.\frac{2a+(\nu+2)x}{x(a+x)}+\frac{(\nu+1)ydx}{x(a+x)}=0$$

quae multiplicata per $xx(a+x)^{\nu+2}\left(\frac{dy}{dx}+\tfrac{1}{2}y\left(\frac{2}{x}+\frac{\nu}{a+x}\right)\right)$ dabit integrale:

$$\tfrac{1}{2}xx(a+x)^{\nu+2}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{2}{x}+\frac{\nu}{a+x}\right)+yy\left(\frac{1}{xx}+\frac{\nu}{x(a+x)}\right)\right)=\text{Const.}$$

Exemplum 2.

888. Sit $K=x^m(aa+xx)^n$ et $L=x^\mu(aa+xx)^\nu$ erit $\frac{dK}{Kdx}=\frac{m}{x}+\frac{2nx}{aa+xx}$ et $\frac{dL}{Ldx}=\frac{\mu}{x}+\frac{2\nu x}{aa+xx}$ et aequatio differentio-differentialis hanc induet formam

$$\frac{ddy}{dx}+dy\left(\frac{m}{x}+\frac{2nx}{aa+xx}\right)+\tfrac{1}{2}ydx\left\{\begin{matrix}\frac{\mu(2m-\mu-2)}{2xx}+\frac{2n\mu+2\nu(m-\mu+1)}{aa+xx}+\frac{2\nu(2n-\nu-2)xx}{(aa+xx)^2}\\+\frac{2Cx^{2\mu-2m}}{(aa+xx)^{2n-2\nu}}\end{matrix}\right\}=0$$

cuius in $x^{2m-\mu}(aa+xx)^{2n-\nu}\left(\frac{dy}{dx}+\tfrac{1}{2}y\left(\frac{\mu}{x}+\frac{2\nu x}{aa+xx}\right)\right)$

ductae

ductae integrale erit

$$\tfrac{1}{2}x^{2m-\mu}(aa+xx)^{2n-\nu}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{\mu}{x}+\frac{2\nu x}{aa+xx}\right)+\frac{1}{4}yy\left(\left(\frac{\mu}{x}+\frac{2\nu x}{aa+xx}\right)^2+\frac{4Cx^{2\mu-2m}}{(aa+xx)^{2n-2\nu}}\right)\right)=\text{Const.}$$

Euoluamus hic casus, quibus aequatio differentio-differentialis hanc obtinet formam:

$$\frac{ddy}{dx}+dy\left(\frac{m}{x}+\frac{2nx}{aa+xx}\right)+\tfrac{1}{4}ydx\left(D+\frac{E}{xx}+\frac{F}{aa+xx}+\frac{Gxx}{(aa+xx)^2}\right)=0.$$

I. Sumatur $\mu=m$ et $\nu=n$, eritque
$D=2C$; $E=\frac{1}{2}m(m-2)$; $F=2n(m+1)$ et $G=2n(n-2)$.

II. Sumatur $\mu=m-1$ et $\nu=n$, eritque
$D=0$; $E=2C+\frac{1}{2}(m-1)^2$; $F=2n(m+1)$ et
$G=2n(n-2)$.

III. Sumatur $\mu=m-1$ et $2n-2\nu=-1$ seu $\nu=n+\frac{1}{2}$ erit vltimus terminus $\frac{2C(aa+xx)}{xx}=2C+\frac{2Caa}{xx}$. Ergo
$D=2C$; $E=2Caa+\frac{1}{2}(m-1)^2$; $F=2(mn+n+1)$;
$G=\frac{1}{2}(2n+1)(2n-5)$.

IV. Sumatur $\mu=m$ et $2n-2\nu=1$ seu $\nu=n-\frac{1}{2}$ erit vltimus terminus $\frac{2C}{aa+xx}$ ideoque
$D=0$; $E=\frac{1}{2}m(m-2)$; $F=2C+2mn+2n-1$;
$G=\frac{1}{2}(2n-1)(2n-3)$.

V. Sumatur $\mu=m+1$ et $\nu=n-\frac{1}{2}$ erit vltimus terminus $\frac{2Cxx}{aa+xx}=2C-\frac{2Caa}{aa+xx}$, ideoque
$D=2C$; $E=\frac{1}{2}(m+1)(m-3)$; $F=-2Caa+2n(m+1)$;
$G=\frac{1}{2}(2n-1)(2n-3)$.

VI.

VI. Sit $\mu = m - 1$ et $\nu = n - \frac{1}{2}$ erit vltimus terminus $\frac{2C}{xx(aa+xx)} = \frac{2C}{aaxx} - \frac{2C}{aa(aa+xx)}$ vnde fit

$$D = 0;\ E = \frac{2C}{aa} + \frac{1}{2}(m-1)^2;\ F = \frac{-2C}{aa} + 2mn + 2n - 2;$$
$$G = \frac{1}{2}(2n-1)(2n-3).$$

VII. Sit $\mu = m + 1$ et $2n - 2\nu = 2$ seu $\nu = n - 1$ erit terminus vltimus $\frac{2Cxx}{(aa+xx)^2}$ ideoque

$$D = 0;\ E = \frac{1}{2}(m+1)(m-3);\ F = 2n(m+1);$$
$$G = 2C + 2(n-1)^2.$$

VIII. Sit $\mu = m + 2$ et $\nu = n - 1$ erit terminus vltimus $\frac{2Cx^4}{(aa+xx)^2} = 2C - \frac{2Caa}{aa+xx} - \frac{2Caaxx}{(aa+xx)^2}$ hincque

$$D = 2C;\ E = \frac{1}{2}(m+2)(m-4);\ F = -2Caa + 2mn + 2n + 2$$
$$\text{et } G = -2Caa + 2(n-1)^2.$$

IX. Sit $\mu = m$ et $\nu = n - 1$ erit terminus vltimus $\frac{2C}{(aa+xx)^2} = \frac{2C}{aa(aa+xx)} - \frac{2Cxx}{a^2(aa+xx)^2}$ hincque

$$D = 0;\ E = \frac{1}{2}m(m-2);\ F = \frac{2C}{aa} + 2mn + 2n - 2;$$
$$G = \frac{-2C}{aa} + 2(n-1)^2.$$

X. Sit $\mu = m - 1$ et $\nu = n - 1$ erit terminus vltimus $\frac{2C}{xx(aa+xx)^2} = \frac{2C}{a^4xx} - \frac{4C}{a^4(aa+xx)} + \frac{2Cxx}{a^4(aa+xx)^2}$, hincque

$$D = 0;\ E = \frac{2C}{a^4} + \frac{1}{2}(m-1)^2;\ F = \frac{-4C}{a^4} + 2mn + 2n - 4;$$
$$G = \frac{2C}{a^4} + 2(n-1)^2.$$

Problema 110.

889. Sumto elemento dx constante, si K et L denotent functiones quascunque ipsius x, inuenire integrale completum huius aequationis differentio-differentialis

$$\frac{ddy}{dx} + \frac{dy}{dx} \cdot \frac{dK}{K} + \tfrac{1}{2}y\left(d.\frac{dL}{Ldx} - \frac{dL^2}{2LLdx} + \frac{dKdL}{KLdx} + \frac{2CLLdx}{KK}\right) = 0.$$

Solutio.

Quoniam haec aequatio integrabilis redditur si multiplicetur per $\frac{KK}{L}\left(\frac{dy}{dx} + \frac{ydL}{2Ldx}\right)$ eius integrale completum, vt supra vidimus est

$$\frac{KK}{2L}\left(\left(\frac{dy}{dx} + \frac{ydL}{2Ldx}\right)^2 + \frac{CLL}{KK}yy\right) = Const.$$

quam aequationem differentialem primi gradus adhuc integrari oportet, quod cum ob constantem indefinitam maxime sit difficile, ea neglecta primo saltem integrale particulare inuestigemus. Erit ergo ex aequatione

$$\left(\frac{dy}{dx} + \frac{ydL}{2Ldx}\right)^2 + \frac{CLL}{KK}yy = 0$$

radicem extrahendo

$$\frac{dy}{dx} + \frac{ydL}{2Ldx} = \frac{Ly}{K}\sqrt{-C} \text{ seu } \frac{dy}{y} + \frac{dL}{2L} = \frac{Ldx}{K}\sqrt{-C}$$

vnde fit

$$y\sqrt{L} = \alpha e^{\int \frac{Ldx}{K}\sqrt{-C}}.$$

Cum

Cum igitur aequationi differentio - differentiali propositae satisfaciant hi duo valores:

$$y = \frac{\alpha}{\sqrt{L}} e^{\int \frac{L\,dx}{K} \sqrt{-C}} \quad \text{et} \quad y = \frac{\beta}{\sqrt{L}} e^{-\int \frac{L\,dx}{K} \sqrt{-C}}$$

bini coniuncti etiam satisfacient, quibus quoniam duae constantes arbitrariae introducuntur, eius integrale completum erit

$$y = \frac{\alpha}{\sqrt{L}} e^{\int \frac{L\,dx}{K} \sqrt{-C}} + \frac{\beta}{\sqrt{L}} e^{-\int \frac{L\,dx}{K} \sqrt{-C}}$$

quae expressio valet si $\sqrt{-C}$ fuerit quantitas realis, sin autem sit imaginaria, erit

$$y = \frac{\gamma}{\sqrt{L}} \sin.\left(\int \frac{L\,dx}{K} \sqrt{C} + \zeta\right)$$

sicque habetur integrale completum aequationis differentio - differentialis propositae.

Coroll. 1.

890. Hinc igitur aequationis differentialis primi gradus

$$\left(\frac{dy}{dx} + \frac{y\,dL}{2L\,dx}\right)^2 + \frac{CLLy^2}{KK} = \frac{AL}{KK}$$

quae per se satis est difficilis, integrale assignare valemus, quod est

$$y = \frac{\alpha}{\sqrt{L}} e^{\int \frac{L\,dx}{K} \sqrt{-C}} + \frac{\beta}{\sqrt{L}} e^{-\int \frac{L\,dx}{K} \sqrt{-C}},$$

si modo debita relatio constantium α et β respectu constantis A definiatur.

Coroll. 2.

891. Erit autem per $\sqrt{L}$ multiplicando et differentiando

$$dy\sqrt{L}+\frac{y\,dL}{2\sqrt{L}}=\frac{\alpha L\,dx}{K}\sqrt{-C}.e^{\int\frac{L\,dx}{K}\sqrt{-C}}-\frac{\beta L\,dx}{K}\sqrt{-C}.e^{-\int\frac{L\,dx}{K}\sqrt{-C}}.$$

Hinc

$$\frac{dy}{dx}+\frac{y\,dL}{2L\,dx}=\frac{\sqrt{-CL}}{K}\left(\alpha e^{\int\frac{L\,dx}{K}\sqrt{-C}}-\beta e^{-\int\frac{L\,dx}{K}\sqrt{-C}}\right)$$

vnde fit

$$\frac{AL}{KK}=\frac{-CL}{KK}\left(\alpha e^{\int\frac{L\,dx}{K}\sqrt{-C}}-\beta e^{-\int\frac{L\,dx}{K}\sqrt{-C}}\right)^2+\frac{CL}{KK}\left(\alpha e^{\int\frac{L\,dx}{K}\sqrt{-C}}+\beta e^{-\int\frac{L\,dx}{K}\sqrt{-C}}\right)^2$$

ideoque $A=4C\alpha\beta$ seu $\beta=\frac{A}{4C\alpha}$.

Scholion 1.

892. Quamuis ergo aequationem propositam ope idonei multiplicatoris integrare licuerit, altera tamen integratio maximis difficultatibus premi videbatur. Interim tamen ope substitutionis aequatio illa differentialis primi gradus tractatu facilis redditur; posito enim $y=\frac{z}{\sqrt{L}}$, vt sit $dy\sqrt{L}+\frac{y\,dL}{2\sqrt{L}}=dz$ oritur $\left(\frac{dz}{dx\sqrt{L}}\right)^2+\frac{CLzz}{KK}=\frac{AL}{KK}$, hinc $\frac{dz}{dx}=\frac{L}{K}\sqrt{(A-Czz)}$ seu $\frac{dz}{\sqrt{(A-Czz)}}=\frac{L\,dx}{K}$ quae integrata dat

$$l(z\sqrt{-C}+\sqrt{(A-Czz)})=\int\frac{L\,dx}{K}\sqrt{-C}+lB$$

vnde praecedens integrale eruitur. Caeterum forma nostrae aequationis differentio-differentialis aliquanto commodius exhiberi potest hoc modo: Si P et R sint

sint functiones quaecunque ipsius x, sumaturque elementum dx constans, huius aequationis

$$ddy - dy\left(\frac{dP}{P} + \frac{dR}{R}\right) - y\left(d.\frac{dP}{P} - \frac{dPdR}{PR} + \frac{aaRR}{PP}dx^2\right) = 0$$

bis integratae integrale completum est:

$$y = \alpha P e^{a\int\frac{Rdx}{P}} + \beta P e^{-a\int\frac{Rdx}{P}}$$

siquidem a sit quantitas realis. At si $a = 0$ erit $y = P(\alpha + \beta\int\frac{Rdx}{P})$, sin autem sit $aa = -cc$ erit $y = \alpha P$ sin $(\beta + c\int\frac{Rdx}{P})$. Tum vero illa aequatio integrabilis redditur si multiplicetur per $\frac{1}{RRdx^2}(dy - \frac{ydP}{P})$ eritque integrale primum:

$$\frac{1}{2RRdx^2}\left(\left(dy - \frac{ydP}{P}\right)^2 - \frac{aaRR}{PP}y^2dx^2\right) = \text{Const.}$$

Hinc patet in illa aequatione differentio-differentiali commode hanc substitutionem adhiberi $y = Pz$, qua ea transformatur in

$$ddz + dz\left(\frac{dP}{P} - \frac{dR}{R}\right) - \frac{aaRR}{PP}zdx^2 = 0,$$

quae per $\frac{PPdz}{RRdx^2}$ multiplicata sponte fit integrabilis. Quin etiam posito $\frac{P}{R} = S$, vt habeatur

$$ddz + \frac{dSdz}{S} - \frac{aazdx^2}{SS} = 0$$

multiplicator $\frac{SSdz}{dx^2}$, statim dat integrale

$$\frac{SSdz^2}{2dx^2} - \tfrac{1}{2}aazz = \text{Const.}$$

Scholion 2.

893. Viciſſim ergo ex hac forma ſimpliciſſima

$$SSddz+SdSdz-aazdx^2=0$$

quae per dz multiplicata integrabilis redditur, formas magis complicatas deriuare potuiſſemus, ponendo $z=\frac{y}{p}$ et $S=\frac{P}{R}$. Quae quanquam in formis generalibus ſatis perſpicua, tamen in exemplis determinatis plerumque haec deriuatio nimis eſt occulta, quam vt menti occurrere poſſit. Veluti in caſibus §. 888. euolutis, ſi N°. IX. ſumamus $m=2$, et $C=(n-1)^2aa$, fiet $D=0$, $E=0$, $F=2n(n+1)$ et $G=0$, vnde habetur haec aequatio:

$$\frac{ddy}{dx}+2dy\left(\frac{1}{x}+\frac{nx}{aa+xx}\right)+\frac{n(n+1)ydx}{aa+xx}=0 \quad \text{ſeu}$$

$$ddy+\frac{2dxdy(aa+(n+1)xx)}{x(aa+xx)}+\frac{n(n+1)ydx^2}{aa+xx}=0$$

quae integrabilis redditur ope multiplicatoris

$$xx(aa+xx)^{n+1}\left(\frac{dy}{dx}+\frac{y(aa+nxx)}{x(aa+xx)}\right),$$

integrali exiſtente

$$\tfrac{1}{2}xx(aa+xx)^{n+1}\left(\left(\frac{dy}{dx}+\frac{y(aa+nxx)}{x(aa+xx)}\right)^2+\frac{(n+1)^2aayy}{(aa+xx)^2}\right)=\text{Conſt.}$$

Pro integrali ergo particulari erit

$$\frac{dy}{y}+\frac{dx}{x}+\frac{(n-1)xdx}{aa+xx}=\pm\frac{(n-1)adx\sqrt{-1}}{aa+xx}$$

vnde colligitur

$$xy(aa+xx)^{\frac{n-1}{2}}=\alpha\left(\frac{a+x\sqrt{-1}}{a-x\sqrt{-1}}\right)^{\pm\frac{(n-1)}{2}}.$$

Ergo

Ergo bina integralia particularia coniuncta dant

$$y = \frac{\alpha}{x}(a - x\sqrt{-1})^{-n+1} + \frac{\beta}{x}(a + x\sqrt{-1})^{-n+1}$$

integrale completum. Hoc antem casu aequatio nostra ad formam simplicissimam reducetur ope substitutionis $y = \frac{z}{x}(aa + xx)^{\frac{1-n}{2}}$, cuius ratio et inuentio difficilius perspicitur.

Problema III.

894. Sumto elemento dx constante inuestigare conditiones quibus aequatio differentio - differentialis

$$ddy + Pdxdy + Qydx^2 = 0$$

integrabilis redditur, ope multiplicatoris huius formae

$$\frac{ydx^2}{Ldy^2 + Mydydx + Nyydx^2}$$

denotantibus litteris L, M, N, P, Q functiones ipsius x.

Solutio.

Tribuatur denominatori huius fractionis talis forma $(dy + Rydx)(dy + Sydx)$, ac leui attentione adhibita patet integrale huiusmodi formam esse habiturum:

$$V + l\frac{dy + Rydx}{dy + Sydx} = \text{Const.}$$

cuius ergo differentiale aequationem propositam producere debet. Dat autem differentiatio:

$$dV + \frac{(S-R)ydxddy + (R-S)dxdy^2 + ydxdy(dR - dS) + yydx^2(SdR - RdS)}{(dy + Rydx)(dy + Sydx)} = 0$$

quae

quae ad communem denominatorem reducta abit in:

$$\left.\begin{matrix}(S-R)ydxddy+(R-S)dxdy^2+ydxdy(dR-dS)+yydx^2(SdR-RdS)\\ +dVdy^2+(R+S)ydxdydV+RSyydx^2dV\end{matrix}\right\}=0$$

Statuatur $dV=(S-R)dx$, vt aequatio per y diuisibilis euadat, sicque orietur haec aequatio:

$$\left.\begin{matrix}(S-R)ddy+dy(dR-dS)\quad+ydx(SdR-RdS)\\ +(SS-RR)dxdy+RS(S-R)ydx^2\end{matrix}\right\}=0$$

quae vt cum forma proposita conueniat, fieri oportet:

$$P=(R+S)+\frac{dR-dS}{(S-R)dx} \text{ et } Q=RS+\frac{SdR-RdS}{(S-R)dx}$$

quos valores si functiones P et Q habuerint, aequatio

$$ddy+Pdxdy+Qydx^2=0$$

per $\frac{(S-R)ydx}{(dy+Rydx)(dy+Sydx)}$ multiplicata integrale dabit:

$$\int(S-R)dx+l\frac{dy+Rydx}{dy+Sydx}=\text{Const.}$$

Si ponamus $S=M+N$ et $R=M-N$ erit

$$P=2M-\frac{dN}{Ndx} \text{ et } Q=MM-NN+\frac{dM}{dx}-\frac{MdN}{Ndx}.$$

Coroll. 1.

895. Quaecunque ergo functiones ipsius x loco M et N assumantur, indeque definiantur:

$$P=2M-\frac{dN}{Ndx} \text{ et } Q=MM-NN+\frac{dM}{dx}-\frac{MdN}{Ndx}$$

huius aequationis

$$ddy+Pdxdy+Qydx^2=0$$

inte-

integrale erit

$$2\int N dx + l \frac{dy + (M-N) y dx}{dy + (M+N) y dx} = \text{Const.}$$

Coroll. 2.

896. Si ponatur $y = e^{\int z dx}$ fiet nostra aequatio differentialis primi gradus:

$$dz + zz dx + Pz dx + Q dx = 0$$

cuius propterea integrale erit

$$2\int N dx + l \frac{z + (M-N)}{z + (M+N)} = \text{Const.}$$

Coroll. 3.

897. Si velimus vt sit $P = 0$, et aequatio habeatur huiusmodi $ddy + Qydx^2 = 0$, capi debet $2M = \frac{dN}{N dx}$ eritque $Q = \frac{dn}{dx} - MM - NN$, eiusque aequatio integralis

$$2\int N dx + l \frac{dy + (M-N) y dx}{dy + (M+N) y dx} = \text{Const.}$$

Coroll. 4.

898. In genere autem prout constans capiatur vel $+\infty$ vel $-\infty$, obtinebitur integrale particulare vel $dy + (M+N) y dx = 0$ vel $dy + (M-N) y dx = 0$, vnde erit vel

$$y = \alpha e^{-\int (M+N) dx} \text{ vel } y = \beta e^{-\int (M-N) dx}$$

ex quibus nostrae aequationis colligitur integrale completum:

$$y = e^{-\int M dx} (\alpha e^{-\int N dx} + \beta e^{\int N dx}).$$

 Exem-

Exemplum 1.

899. Sit $M=\alpha$ et $N=\beta$, erit $P=2\alpha$ et $Q=\alpha\alpha-\beta\beta$ unde huius aequationis

$$ddy+2\alpha dxdy+(\alpha\alpha-\beta\beta)ydx^2=0$$

integrale est:

$$2\beta x+l\frac{dy+(\alpha-\beta)ydx}{dy+(\alpha+\beta)ydx}=\text{Const.}$$

In quantitatibus autem finitis integrale completum:

$$y=e^{-\alpha x}(Ae^{-\beta x}+Be^{\beta x}).$$

Casu autem quo $\beta\beta=-\gamma\gamma$ aequatio

$$ddy+2\alpha dxdy+(\alpha\alpha+\gamma\gamma)ydx^2=0$$

bis integrata dat:

$$y=Ae^{-\alpha x}\sin.(\gamma x+C)$$

at si $\gamma=0$, aequationis

$$ddy+2\alpha dxdy+\alpha\alpha ydx^2=0$$

integrale est

$$y=e^{-\alpha x}(A+Bx).$$

Exemplum 2.

900. Si $M=\frac{\alpha}{x}$ et $N=\beta x^n$ erit $P=\frac{2\alpha-n}{x}$, et $Q=\frac{\alpha\alpha}{xx}-\beta\beta x^{2n}-\frac{\alpha}{xx}-\frac{\alpha n}{xx}=\frac{\alpha(\alpha-n-1)}{xx}-\beta\beta x^{2n}$.
Ergo huius aequationis

$$ddy+\frac{(2\alpha-n)dxdy}{x}+\frac{\alpha(\alpha-n-1)ydx^2}{xx}-\beta\beta x^{2n}ydx^2=0$$

inte-

Integrale primum eſt

$$\frac{2\beta}{n+1}x^{n+1}+l\frac{x\,dy+(\alpha-\beta x^{n+1})y\,dx}{x\,dy+(\alpha+\beta x^{n+1})y\,dx}=\text{Conſt.}$$

Integrale autem ſecundum

$$y=x^{-\alpha}(\mathrm{A}e^{\frac{-\beta x^{n+1}}{n+1}}+\mathrm{B}e^{\frac{\beta x^{n+1}}{n+1}})$$

ſi $\beta=0$ erit id $y=x^{-\alpha}(\mathrm{A}+\mathrm{B}x^{n+1})$

ſin autem $\beta\beta=-\gamma\gamma$ erit:

$$y=\mathrm{A}x^{-\alpha}\text{ſin.}(\frac{\gamma}{n+1}x^{n+1}+\mathrm{C}.)$$

Coroll. 1.

901. Sumto $n=2\alpha$, vt habeatur haec aequatio

$$ddy-\frac{\alpha(\alpha+1)y\,dx^2}{xx}-\beta\beta x^{4\alpha}y\,dx^2=0$$

erit eius integrale completum

$$\mathrm{N}=x^{-\alpha}(\mathrm{A}e^{\frac{-\beta}{2\alpha+1}x^{2\alpha+1}}+\mathrm{B}e^{\frac{\beta}{2\alpha+1}x^{2\alpha+1}})$$

ſi ſit $\beta=0$ erit id $y=x^{-\alpha}(\mathrm{A}+\mathrm{B}x^{2\alpha+1})$

at ſi $\beta\beta=-\gamma\gamma$ erit hoc integrale

$$y=\mathrm{A}x^{-\alpha}\text{ſin.}(\frac{\gamma}{2\alpha+1}x^{2\alpha+1}+\mathrm{C}).$$

Coroll. 2.

902. Ponamus $\alpha = -1$, vt habeamus hanc aequationem $ddy - \frac{\beta\beta y dx^2}{x^4} = 0$, cuius ergo integrale erit $y = x(Ae^{\frac{\beta}{x}} + Be^{-\frac{\beta}{x}})$, vbi notandum si sit $\beta\beta = -\gamma\gamma$ fore $y = Ax \sin.(\frac{\gamma}{x} + C)$.

Exemplum 3.

903. Ponatur $N = \frac{Ax^m}{\alpha + \beta x^n}$, vt sit $\frac{dN}{Ndx} = \frac{m}{x} - \frac{\beta n x^{n-1}}{\alpha + \beta x^n}$ et sumatur $M = \frac{m}{2x} - \frac{\beta n x^{n-1}}{2(\alpha + \beta x^n)}$ vt fiat $P = 0$ et

$$Q = \frac{-m}{2xx} - \frac{\beta n(n-1)x^{n-2}}{2(\alpha+\beta x^n)} + \frac{\beta\beta nn x^{2n-2}}{2(\alpha+\beta x^n)^2}$$
$$- \frac{mm}{4xx} + \frac{\beta mn x^{n-2}}{2(\alpha+\beta x^n)} - \frac{\beta\beta nn x^{2n-2}}{4(\alpha+\beta x^n)^2}$$
$$- \frac{AAx^{2m}}{(\alpha+\beta x^n)^2}$$

siue

$$Q = \frac{-m(m+2)}{4xx} + \frac{n(m-n+1)\beta x^{n-2}}{2(\alpha+\beta x^n)} + \frac{\beta\beta nn x^{2n-2} - 4AAx^{2m}}{4(\alpha+\beta x^n)^2}$$

et ob $\int M dx = \frac{1}{2} l N$ erit integrale:

$$y = \frac{1}{\sqrt{N}}(Ce^{-\int N dx} + De^{\int N dx})$$

huius aequationis $ddy + Qy dx^2 = 0$.

Vt

Vt expressio ipsius Q fiat simplicior, hoc fieri potest pluribus modis, dum numerator partis postremae per $\alpha + \beta x^n$ diuisibilis redditur.

I. Sit $m = n - 1$ et $AA = \frac{1}{4}\beta\beta nn$, eritque $Q = -\frac{(nn-1)}{4xx}$ tum vero $N = \frac{\frac{1}{2}\beta n x^{n-1}}{\alpha + \beta x^n}$ et $\int N dx = \frac{1}{2} l(\alpha + \beta x^n)$ vnde aequationis $ddy - \frac{(nn-1) y dx^2}{4xx} = 0$ integrale est

$$y = \frac{1}{\sqrt{x^{n-1}}}(C + D\alpha + D\beta x^n).$$

II. Sit $2m = -2$ seu $m = -1$ et $4AA = \alpha\alpha nn$ erit

$$Q = \frac{1}{4xx} - \frac{nn\beta x^{n-2}}{2(\alpha + \beta x^n)} + \frac{\beta nn x^{n-2} - \alpha nn x^{-2}}{4(\alpha + \beta x^n)} \text{ seu}$$

$Q = \frac{1-nn}{4xx}$ vt ante.

III. Sit $2m = -n - 2$ seu $m = \frac{-n-2}{2}$ et $4AA = -\frac{\alpha^2 nn}{\beta}$ fietque

$$Q = \frac{-(nn-4)}{16xx} - \frac{3nn\beta x^{n-2}}{4(\alpha + \beta x^n)} + \frac{nn(\beta\beta x^{n-2} - \alpha\beta x^{-2} + \alpha\alpha x^{-n-2})}{4\beta(\alpha + \beta x^n)}$$

seu $$Q = \frac{4 - nn}{16xx} + \frac{nn(\alpha\alpha x^{-n-2} - \alpha\beta x^{-2} - 2\beta\beta x^{n-2})}{4\beta(\alpha + \beta x^n)}$$

quae expressio abit in

$$Q = \frac{4-nn}{16xx} + \frac{nn}{4\beta}(\alpha x^{-n-2} - 2\beta x^{-2}) = \frac{4-9nn}{16xx} + \frac{nn\alpha}{4\beta x^{n+2}}.$$

Quare cum fit

$$N = + \frac{n\alpha\sqrt{-\alpha}}{2\sqrt{\beta}} \cdot \frac{x^{-\frac{n+2}{2}}}{\alpha + \beta x^n} \text{ erit}$$

$$\int N\,dx = \frac{n\alpha\sqrt{-\alpha}}{2\sqrt{\beta}} \int \frac{dx}{(\alpha + \beta x^n)\, x^{\frac{n+2}{2}}},$$

fumatur $n = \frac{2}{3}$, vt fiat $m = -\frac{4}{3}$ et $Q = \frac{\alpha}{9\beta x^{\frac{8}{3}}}$ et

$$N = \frac{\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \cdot \frac{x^{-\frac{4}{3}}}{\alpha + \beta x^{\frac{2}{3}}} \text{ hinc } \int N\,dx = \frac{\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \int \frac{dx}{(\alpha + \beta x^{\frac{2}{3}})\, x^{\frac{4}{3}}}$$

ficque aequationis $ddy + \frac{\alpha}{9\beta x^{\frac{8}{3}}}\, y\,dx^2 = 0$ integrale erit

$$y = \frac{1}{\sqrt{N}}\left(C e^{-\int N\,dx} + D e^{\int N\,dx}\right).$$

Sin autem capiatur $n = -\frac{2}{3}$ vt fiat $m = -\frac{2}{3}$ et $Q = \frac{\alpha}{9\beta x^{\frac{4}{3}}}$

erit $N = \frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \cdot \frac{x^{-\frac{2}{3}}}{\alpha + \beta x^{-\frac{2}{3}}}$ et

$$\int N\,dx = \frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \int \frac{x^{\frac{1}{3}}\,dx}{\alpha x + \beta x^{\frac{1}{3}}} = \frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \int \frac{dx}{\alpha x^{\frac{2}{3}} + \beta};$$

vnde aequatio;

$$ddy + \frac{\alpha y\,dx^2}{9\beta x^{\frac{4}{3}}} = 0$$

fimili modo integratur.

Scho-

Scholion 1.

904. Aequationem ergo $ddy + Ax^m y dx^2 = 0$ his casibus integrare licuit $m = 0$, $m = -4$, $m = -\frac{4}{3}$, $m = -\frac{8}{3}$ et $m = -2$, seu $m = -2 \pm \frac{2}{1}$, $m = -2 \pm \frac{2}{3}$. Quodsi vlterius ponamus $N = \frac{Ax^\lambda}{\alpha + \beta x^n + \gamma x^{2n}}$, simili modo integrationem casuum istius aequationis $m = -2 \pm \frac{2}{5}$ impetrabimus, quibus quoque aequatio differentialis primi gradus

$$dz + zzdx + Ax^m dx = 0$$

integrationem admittit. Haec autem casuum integrabilium inuestigatio nimis est operosa, quam vt eam fusius prosequamur, praesertim cum infra methodus occurrat haec omnia commodius euoluendi.

Scholion 2.

905. Ex his colligere licet, quantus fructus ex inuentione multiplicatorum, quibus etiam aequationes differentio-differentiales integrabiles redduntur, expectari queat, etiamsi exempla hic tractata tantum leue huius methodi specimen referant. Aliquas autem saltem multiplicatorum formas hic sum contemplatus, neque vllum est dubium, quin plures aliae formae pari successu in vsum vocari queant. In hoc porro capite tantum eiusmodi aequationes differentio-differentiales tractauimus, in quibus altera variabilis y cum suis differentialibus dy et ddy vbique

vbique vnicam obtinet dimenſionem. Verum eadem methodus quoque ad alia huiusmodi aequationum genera extenditur, quae etſi parum adhuc eſt exculta, tamen vſu non carebit ſequens applicatio, vbi integratio aliarum aequationum differentialium ſecundi gradus, quae aliis methodis tractatu difficillimae videantur, ope multiplicatorum docebitur.

CAPVT VI.

CAPVT VI.

DE INTEGRATIONE ALIARVM AEQVATIONVM DIFFERENTIO-DIFFERENTIALIVM PER IDONEOS MVLTIPLICATORES INSTITVENDA.

Problema 112.

906.

Posito elemento dx constante si proposita sit huiusmodi aequatio:

$$ddy + \frac{Aydx^2}{(Byy+C+2Dx+Exx)^2} = 0,$$

inuenire multiplicatorem quo ea integrabilis reddatur.

Solutio.

Tentetur talis multiplicatoris forma $2Pdy + 2Qydx$ vbi P et Q sint functiones ipsius x, et producti

$$2ddy(Pdy + Qydx) + \frac{2Aydx^2(Pdy + Qydx)}{(Byy+C+2Dx+Exx)^2} = 0$$

integrale statuatur:

$$Pdy^2 + 2Qydxdy + Vdx^2 = \text{Const.}\, dx^2$$

 vbi

vbi V ſit functio binas variabiles x et y complectens. Erit ergo facta aequalitate:

$$\begin{matrix} +dP\,dy^2 & +2y\,dx\,dQ\,dy \\ +2Q\,dx\,dy^2 & +dx^2\,dV \end{matrix} - \frac{2Ay\,dx^2(P\,dy+Qy\,dx)}{(Byy+C+2Dx+Exx)^2} = 0$$

quae per integrationem valorem ipſius V ſuppeditare nequit, niſi ſit $dP+2Q\,dx=0$, eritque tum:

$$dV = \frac{Ay(2P\,dy-y\,dP)}{(Byy+C+2Dx+Exx)^2} - 2y\,dy.\frac{dQ}{dx}$$

cuius formulae, ſiquidem integrationem admittat, integrale ex variabilitate ipſius y erit

$$\frac{-AP}{B(Byy+C+2Dx+Exx)} - yy.\frac{dQ}{dx} = V.$$

Sumto ergo y conſtante neceſſe eſt ſit

$$-\frac{A\,dP(Byy+C+2Dx+Exx)+2AP\,dx(D+Ex)}{B(Byy+C+2Dx+Exx)^2} - yy.\frac{ddQ}{dx} = \frac{-Ayy\,dP}{(Byy+C+2Dx+Exx)^2},$$

cui ſatisfit ſi $\frac{ddQ}{dx^2}=0$ et

$$-dP(C+2Dx+Exx)+2P\,dx(D+Ex)=0$$

quae duae conditiones an ſimul conſiſtere poſſint videndum eſt. Poſterior autem dat:

$$\frac{dP}{P} = \frac{2D\,dx+2Ex\,dx}{C+2Dx+Exx} \text{ ideoque } P = C+2Dx+Exx$$

vnde fit

$$Q = -\frac{dP}{2dx} = -D-Ex, \text{ hinc } \frac{dQ}{dx} = -E \text{ et } \frac{ddQ}{dx^2} = 0.$$

Multiplicator ergo quaeſitus eſt:

$$2dy(C+2Dx+Exx) - 2y\,dx(D+Ex)$$

hinc-

hincque obtinetur integrale:

$$\frac{dy^2}{dx^2}(C+2Dx+Exx)-\frac{2ydy}{dx}(D+Ex)$$
$$\frac{-A(C+2Dx+Exx)}{B(Byy+C+2Dx+Exx)}+Eyy=\text{Const.}$$

seu vtrinque addendo $\frac{A}{B}$

$$\frac{dy^2}{dx^2}(C+2Dx+Exx)-\frac{2ydy}{dx}(D+Ex)+\frac{Ayy}{Byy+C+2Dx+Exx}+Eyy=\text{Const.}$$

Coroll. 1.

907. Haec ergo aequatio $ddy+\frac{aaydx^2}{(yy+xx)^2}=0$, vbi $A=aa$, $B=1$, $C=0$, $D=0$ et $E=1$, integrabilis redditur multiplicatore $2xxdy-2yxdx$ eiusque integrale erit

$$\frac{xxdy^2}{dx^2}-\frac{2xydy}{dx}+yy+\frac{aayy}{yy+xx}=bb.$$

Coroll. 2.

908. Si hic ponatur $y=ux$, ob $dy=udx+xdu$ habebimus:

$$xxuu+\frac{2ux^3du}{dx}+\frac{x^4du^2}{dx^2}+\frac{aauu}{1+uu}=bb$$
$$-2xxuu-\frac{2ux^3du}{dx}$$
$$+xxuu$$ siue

$\frac{x^4du^2}{dx^2}=\frac{bb+(bb-aa)uu}{1+uu}$ ergo $\frac{dx}{xx}=\frac{du\sqrt{(1+uu)}}{\sqrt{(bb+(bb-aa)uu)}}$ vnde tam x quam y per u determinatur.

Coroll. 3.

909. Simili etiam modo integratio in genere perfici poteſt. Sit enim breuitatis gratia

$$C + 2Dx + Exx = Bzz \text{, erit } D + Ex = \frac{Bz\,dz}{dx}$$

et aequatio noſtra fiet:

$$\frac{Bzz\,dy^2}{dx^2} - \frac{2Byz\,dy\,dz}{dx^2} + Eyy + \frac{Ayy}{B(yy+zz)} = \frac{K}{B}$$

quae poſito $y = uz$ abit in

$$\frac{Bz^4\,du^2}{dx^2} - \frac{Buuzz\,dz^2}{dx^2} + Euuzz + \frac{Auu}{B(1+uu)} = \frac{K}{B}$$

At $\frac{zz\,dz^2}{dx^2} = \frac{(D+Ex)^2}{BB}$, vnde oritur:

$$\frac{Bz^4\,du^2}{dx^2} + \frac{CE-DD}{B}uu = \frac{K+(K-A)uu}{B(1+uu)} \text{ ſeu}$$

$$\frac{BBz^4\,du^2}{dx^2} = \frac{K+(K-A+DD-CE)uu+(DD-CE)u^4}{1+uu}$$

ita vt ſit reſtituto valore ipſius z

$$\frac{dx}{C+2Dx+Exx} = \frac{du\sqrt{(1+uu)}}{\sqrt{(K+(K-A+DD-CE)uu+(DD-CE)u^4)}}$$

ſicque x definitur per u, indeque etiam

$$y = uz = u\sqrt{\frac{C+2Dx+Exx}{B}}.$$

Scholion.

910. Hinc patet ſubſtitutio, qua tam ipſa aequatio differentio-differentialis propoſita, quam multiplicator ad formam commodiorem reduci debet. Poſito enim ad abbreuiandum $C + 2Dx + Exx = Bzz$ aequa-

aequatio nostra $ddy + \frac{Aydx^2}{B^2(yy+zz)^2} = 0$ ope substitutionis $y = uz$ transformatur in

$$zddu + 2dzdu + uddz + \frac{Audx^2}{BBz^3(1+uu)^2} = 0$$

cuius multiplicator est

$$2B(zzdy - yzdz) \text{ seu } 2Bz^3du,$$

vel simpliciter z^3du. Cum autem sit $dz = \frac{dx(D+Ex)}{Bz}$ erit

$$ddz = \frac{Edx^2}{Bz} - \frac{dxdz(D+Ex)}{Bzz} = \frac{Edx^2}{Bz} - \frac{dx^2(D+Ex)^2}{BBz^3} = \frac{(CE-DD)dx^2}{BBz^3}$$

ita vt sit $z^3ddz = \frac{CE-DD}{BB}dx^2$, vnde aequatio nostra per z^3du multiplicata induit hanc formam

$$z^4duddu + 2z^3dzdu^2 + \frac{CE-DD}{BB}ududx^2 + \frac{Adudx^2}{BB(1+uu)^2} = 0$$

manifesto integrabilem, integrali existente:

$$\tfrac{1}{2}z^4du^2 + \frac{CE-DD}{2BB}uudx^2 - \frac{Adx^2}{2BB(1+uu)} = \tfrac{1}{2}\text{Const.}\,dx^2$$

cuius adeo noua integratio ob z functionem ipsius x mox in oculos incurrit, cum sit

$$zzdu = dx\sqrt{\left(\text{Const.} + \frac{DD-CE}{BB}uu + \frac{A}{BB(1+uu)}\right)}$$

vbi variabiles u et x sponte separantur. Caeterum hic notetur functionem pro z assumtam satisfacere aequationi $z^3ddz = \alpha dx^2$, cum tamen eius ratio non sit manifesta. Multiplicando autem hanc aequationem per $\frac{2dz}{z^3}$ prodit $2dzddz = \frac{2\alpha dx^2dz}{z^3}$, cuius integrale est $dz^2 = \beta dx^2 - \frac{\alpha dx^2}{zz}$ seu $dx = \frac{zdz}{\sqrt{(\beta zz-\alpha)}}$

vnde porro fit $\beta x+\gamma=\sqrt{(\beta zz-\alpha)}$, ideoque $\beta zz=\alpha+\gamma\gamma+2\beta\gamma x+\beta\beta xx$, quae eſt ipſa noſtra forma.

Problema 113.

911. Sumto elemento dx conſtante inuenire formam generaliorem aequationum differentio-differentialium quae ope huiusmodi multiplicatoris $Mydx+Ndy$ integrabiles reddantur.

Solutio.

Quia multiplicator ope ſubſtitutionis $y=Ru$ in formam ſimpliciſſimam Sdu transmutari poteſt hac ſubſtitutione ipſa aequatio differentio-differentialis induat hanc formam:

$$ddu+Pdxdu+\frac{Udx^2}{S}=0$$

cuius poſtremum membrum per Sdu multiplicatum ſponte eſt integrabile, ſi quidem U denotet functionem quamcunque ipſius u, dum R et S et P ſint functiones ipſius x. Cum ergo aequatio

$$Sduddu+PSdxdu^2+Udx^2du=0$$

debeat eſſe integrabilis, poſito integrali

$$\tfrac{1}{2}Sdu^2+dx^2\int Udu=\tfrac{1}{2}Cdx^2$$

neceſſe eſt ſit

$$\tfrac{1}{2}dSdu^2=PSdxdu^2 \text{ ſeu } Pdx=\frac{dS}{2S}.$$

Quo-

Quocirca haec forma generalis

$$ddu + \frac{dSdu}{2S} + \frac{Udx^2}{S} = 0$$

per Sdu multiplicata dabit integrale

$$Sdu^2 = dx^2(C - 2\int Udu)$$

quod denuo integratum praebet:

$$\int\frac{dx}{\sqrt{S}} = \int\frac{du}{\sqrt{(C - 2\int Udu)}}.$$

Cum igitur haec ſint manifeſta, ponendo $u = \frac{y}{R}$ ad formas magis intricatas regrediamur, ita vt iam ſit $U =$ funct. $\frac{y}{R}$. Nunc vero eſt

$$du = \frac{dy}{R} - \frac{ydR}{RR} \text{ et } ddu = \frac{ddy}{R} - \frac{2dRdy}{RR} - \frac{yddR}{RR} + \frac{2ydR^2}{R^3}$$

vnde aequatio noſtra fit

$$\frac{ddy}{R} - \frac{2dRdy}{RR} - \frac{yddR}{RR} + \frac{2ydR^2}{R^3} + \frac{Udx^2}{S} = 0$$
$$+ \frac{dSdy}{2RS} \qquad - \frac{ydRdS}{2RRS}$$

quae per $\frac{S}{RR}(Rdy - ydR)$ integrabilis redditur.

Vt igitur ad formam ſupra propoſitam accedamus, ſtatuamus $S = \alpha R^4$, et aequatio

$$\frac{ddy}{R} - \frac{yddR}{RR} + \frac{Udx^2}{\alpha R^4} = 0$$

per $\alpha RR(Rdy - ydR)$ multiplicata integrabilis redditur. Seu haec aequatio

$$Rddy - yddR + \frac{dx^2}{RR}f:\frac{y}{R} = 0$$

per $Rdy - ydR$ multiplicata fit integrabilis.

Vt

Vt via ad integrationem perueniendi magis occultetur, ponatur $f:\frac{y}{R}=\frac{\alpha y}{R}+V$, vt V fit functio homogenea nullius dimenfionis ipfarum y et R, ac ponatur $yddR=\frac{\alpha y dx^2}{R^3}$, vt fiat

$$Rddy+\frac{Vdx^2}{RR}=0 \text{ feu } ddy+\frac{Vdx^2}{R^3}=0$$

quae multiplicatore $R(Rdy-ydR)$ redditur integrabilis. At cum fit $ddR=\frac{\alpha dx^2}{R^3}$, erit vt fupra vidimus $R=\sqrt{(\alpha+2\beta x+\gamma xx)}$, vnde dum V fit functio homogenea nullius dimenfionis ipfarum y et $R=\sqrt{(\alpha+2\beta x+\gamma xx)}$ aequatio

$$ddy+\frac{Vdx^2}{(\alpha+2\beta x+\gamma xx)^{\frac{3}{2}}}=0$$

ope multiplicatoris $(\alpha+2\beta x+\gamma xx)dy-(\beta+\gamma x)ydx$ integrabilis euadit.

Coroll. 1.

912. Pofito autem $R=\sqrt{(\alpha+2\beta x+\gamma xx)}$ aequatio noftra per $RRdy-RydR$ multiplicata fit

$$RRdyddy-RydRddy+\frac{Vdx^2(Rdy-ydR)}{RR}=0$$

cuius integrale eft:

$$\tfrac{1}{2}RRdy^2-RydRdy+\int ydy(RddR+dR^2)+dx^2\int V\,d\tfrac{y}{R}=\text{Conft.}dx^2$$

vbi eft

$$RddR+dR^2=d.RdR=d.(\beta+\gamma x)dx=\gamma dx^2,$$

ficque

ſicque integrale eſt

$$RRdy^2 - 2Ry\,dR\,dy + \gamma yy\,dx^2 + 2dx^2 \int V d\cdot\frac{y}{R} = \text{Conſt.}\,dx^2.$$

Coroll. 2.

913. Quia V eſt functio ipſius $\frac{y}{R}$ formulae $\int V d.\frac{y}{R}$ integrale habetur. Pro vlteriore vero integratione poſito $y = Ru$ et $\int V du = U$, habebitur:

$$R^4 du^2 - RRuu\,dR^2 + \gamma RRuu\,dx^2 + 2U\,dx^2 = G\,dx^2 \text{ ſeu}$$

$$R^4 du^2 = dx^2(G - 2U + (\beta\beta - \alpha\gamma)uu)$$

hincque

$$\frac{dx}{\alpha + 2\beta x + \gamma xx} = \frac{du}{\sqrt{(G - 2U + (\beta\beta - \alpha\gamma)uu)}}$$

ac porro $y = u\sqrt{(\alpha + 2\beta x + \gamma xx)}$.

Scholion.

914. Haec ergo aequatio $ddy + \frac{V\,dx^2}{R^3} = 0$, exiſtente $R = \sqrt{(\alpha + 2\beta x + \gamma xx)}$ multo latius patet ea quam in praecedente problemate tractauimus, propterea quod hic pro V accipere licet functionem quamcunque homogeneam nullius dimenſionis ipſarum y et R. Si enim ſumatur $V = \frac{AR^3 y}{(myy + RR)^2}$, ipſa aequatio primum tractata oritur. Caeterum ex methodo, qua illam aequationem elicuimus apparet, eam per reſtrictionem ad hanc formam occultam eſſe perductam, cum ea aequatio, vnde eſt nata:

$$R\,ddy - y\,ddR + \frac{dx^2}{RR} f : \frac{y}{R} = 0$$

 per-

perſpicue integrationem admittat, ſi per $Rdy-ydR$ multiplicetur. Eſt enim
$Rddy-yddR=d.(Rdy-ydR)$ et $\frac{Rdy-ydR}{RR}=d.\frac{y}{R}$,
vnde facta multiplicatione habebimus:

$$(Rdy-ydR)d.(Rdy-ydR)+dx^2 f:\tfrac{y}{R}.d.\tfrac{y}{R}=0,$$

cuius aequationis vtrumque membrum per ſe eſt integrabile. In aequatione autem inde eruta integrabilitas minus perſpicitur, multo magis integratio eſt abſcondita in aequationibus ſequentibus.

Problema 114.

915. Sumto elemento dx conſtante, integrationem huius aequationis

$$yyddy+ydy^2+Axdx^2=0$$

ope multiplicatoris eam integrabilem reddentis perficere.

Solutio.

Hic fruſtra tentatur multiplicator huius formae $Ldy+Mdx$; tentetur ergo haec forma

$$3Ldy^2+2Mdxdy+Ndx^2$$

ac ponatur producti integrale:

$$Lyydy^3+Myydxdy^2+Nyydx^2dy+Vdx^3=Cdx^3$$

cuius differentiatio perducit ad hanc aequationem

$$\begin{aligned}dx^3dV=\;&3Lydy^4+2Mydxdy^3+Nydx^2dy^2+2AMxdx^3dy+ANxdx^4\\&-2Lydy^4-yydxdy^3\left(\tfrac{dL}{dx}\right)+3ALxdx^2dy^2-yydx^3dy\left(\tfrac{dN}{dx}\right)\\&-yydy^4\left(\tfrac{dL}{dy}\right)-2Mydxdy^3-yydx^2dy^2\left(\tfrac{dM}{dx}\right)\\&-yydxdy^3\left(\tfrac{dM}{dy}\right)-2Nydx^2dy^2\\&-yydx^2dy^2\left(\tfrac{dN}{dy}\right)\end{aligned}$$

quae

quae formula vt integrationem admittat, membra quae dy^4, dy^3 et dy^2 continent, euanescere debent: vnde primo colligitur $L - y(\frac{dL}{dy}) = 0$, vbi $(\frac{dL}{dy})$ nascitur ex differentiatione ipsius L posito x constante. Consideretur ergo x vt quantitas constans, eritque $\frac{dL}{L} = \frac{dy}{y}$ ideoque $L = yf:x$. Negligamus autem hanc functionem ipsius x, seu eius loco vnitatem sumamus, vt sit $L = y$ et $(\frac{dL}{dx}) = 0$ secundo ergo esse debet: $(\frac{dM}{dy}) = 0$. Sumamus igitur $M = 0$, etiamsi M denotare possit functionem quamcunque ipsius x, quandoquidem videbimus hoc modo negotium confici posse. Tertio itaque habebimus:

$$-Ny + 3Axy - yy(\frac{dN}{dy}) = 0$$

sumto ergo x constante erit $3Axdy = Ndy + ydN$, ideoque $Ny = 3Axy$ seu $N = 3Ax$, vbi iterum functionem ipsius x quae loco constantis ingrederetur, negligimus. Cum igitur hactenus inuenimus $L = y$, $M = 0$ et $N = 3Ax$ erit $dV = -3Ayydy + 4AAxxdx$, quae formula cum sponte sit integrabilis scilicet $V = -Ay^3 + AAx^4$, multiplicator nostram aequationem integrabilem reddens erit: $3ydy^2 + 3Axdx^2$, et producti integrale habebitur:

$$y^3dy^3 + 3Axyydx^2dy - Ay^3dx^3 + AAx^4dx^3 = Cdx^3$$

quod ob constantem C est integrale completum.

Coroll. 1.

916. Huius integralis membrum primum commode in tres factores resolui potest. Si ponantur formulae $z^3 - A$ factores $(z-\alpha)(z-\beta)(z-\gamma)$, vt sit $\alpha = \sqrt[3]{A}$, $\beta = \frac{-1+\sqrt{-3}}{2}\sqrt[3]{A}$ et $\gamma = \frac{-1-\sqrt{-3}}{2}\sqrt[3]{A}$, erit integrale inuentum:

$$\left(\frac{ydy}{dx} - \alpha y + \alpha\alpha x\right)\left(\frac{ydy}{dx} - \beta y + \beta\beta x\right)\left(\frac{ydy}{dx} - \gamma y + \gamma\gamma x\right) = C$$

existente

$\alpha + \beta + \gamma = 0$, $\alpha\beta + \alpha\gamma + \beta\gamma = 0$ et $\alpha\beta\gamma = 1$: posito enim $\frac{ydy}{dx} = z$ habetur haec forma

$$z^3 + 3Axyz - Ay^3 + AAx^3$$

cuius factor si ponatur $z - p - q$ fit

$$z^3 - 3pqz - p^3 - q^3 = 0$$

ideoque $p = y\sqrt[3]{A}$ et $q = -x\sqrt[3]{A^2}$.

Coroll. 2.

917. Sumto ergo constante $C = 0$, tria obtinentur integralia particularia; $ydy - \alpha ydx + \alpha\alpha xdx = 0$, et loco α scribendo β et γ,

$ydy - \beta ydx + \beta\beta xdx = 0$ et $ydy - \gamma ydx + \gamma\gamma xdx = 0$ quae posito $y = ux$ dant $\frac{dx}{x} = \frac{-udu}{uu - \alpha u + \alpha\alpha}$ et porro integrando

$$lx = l\frac{\alpha}{\sqrt{(\alpha\alpha - \alpha u + uu)}} - \frac{1}{\sqrt{3}}\,\text{Ang. tang.}\,\frac{u\sqrt{3}}{2\alpha - u} + \text{Const.}$$

Scholion 1.

918. Aequationem autem differentialem primi ordinis inuentam difficile est denuo integrare. A potesta-

teſtatibus quidem differentialium ponendo $dy=pdx$ et $y=ux$, vnde fit $\frac{dx}{x}=\frac{du}{p-u}$ liberari poteſt, prodit enim

$$x^3(u^3p^3+3\,Auup-Au^3+AA)=C$$

quae ſumtis logarithmis differentiata dat:

$$\frac{dx}{x}+\frac{uudp(upp+A)+udu(up^3+2Ap-Au)}{u^3p^3+3Auup-Au^3+AA}=0$$

quae loco $\frac{dx}{x}$ ſcripto $\frac{du}{p-u}$ abit in

$$du(upp+A)^2+uu(p-u)dp(upp+A)=0$$

ac per $upp+A$ diuidendo oritur:

$$A\,du+uppdu+puudp-u^3dp=0$$

quae ponendo $p=\frac{q}{u}$ aliquanto fit ſimplicior ſcilicet:

$$A\,du+qdq+qudu-uudq=0$$

cui autem poſito $A=m^2$ etſi particulariter ſatisfacit $q=mu-mm$, tamen inde integratio completa erui vix poſſe videtur. Caeterum eadem haec aequatio inter p et u immediate elicitur ex aequatione differentio-differentiali propoſita, quoniam in ea binae variabiles x et y vbique eundem dimenſionum numerum conſtituunt. Poſito enim $dy=pdx$ et $y=ux$, abit ea in

$$uuxdp+uppdx+A\,dx=0 \quad \text{ſeu} \quad \frac{dx}{x}=\frac{-uudp}{A+upp}=\frac{du}{p-u}$$

quae eſt ipſa praecedens aequatio.

Scholion 2.

919. Interim tamen aequatio propofita complete integrari poteft, indeque etiam eae, quas ex ea elicuimus. Hoc autem prorfus fingulari ratione praeftatur, aequationem illam adeo ad differentialia tertii ordinis euehendo. Cum enim fit

$$y d.\frac{y\,dy}{dx} + A x\,dx = 0$$

ftatuatur $\frac{dx}{y} = dv$, vt fiat

$$y d.\frac{dy}{dv} + A x\,dx = 0 \text{ feu } d.\frac{dy}{dv} + A x\,dv = 0,$$

quae fumto elemento dv conftante denuo differentiata praebet

$$\frac{d^3 y}{dv} + A\,dx\,dv = 0 \text{ feu } d^3 y + A y\,dv^3 = 0,$$

quae forma ita eft comparata, vt fi ei particulariter fatisfaciant $y = P$, $y = Q$, $y = R$, etiam fatisfaciat $y = DP + EQ + FR$. Iam vero illi fatisfacit $y = e^{-\alpha v}$ fi fuerit $\alpha^3 = A$; cum igitur in Coroll. 1. ternae litterae α, β, γ eadem conditione fint praeditae, habebitur integrale completum

$$y = De^{-\alpha v} + Ee^{-\beta v} + Fe^{-\gamma v}$$

vnde ob $Ax = -\frac{ddy}{dv^2}$ erit

$$x = \frac{-D\alpha\alpha e^{-\alpha v} - E\beta\beta e^{-\beta v} - F\gamma\gamma e^{-\gamma v}}{A}$$

feu

seu mutatis constantibus ob $A = \alpha^3 = \beta^3 = \gamma^3$

$$x = +\mathfrak{A} e^{-\alpha v} + \mathfrak{B} e^{-\beta v} + \mathfrak{C} e^{-\gamma v}$$

$$y = -\mathfrak{A} \alpha e^{-\alpha v} - \mathfrak{B} \beta e^{-\beta v} - \mathfrak{C} \gamma e^{-\gamma v}.$$

Hinc ergo aequationis

$$A\,du + q\,dy + qu\,du - uu\,dq = 0$$

integrale completum his formulis continetur:

$$u = \frac{-\mathfrak{A} \alpha e^{-\alpha v} - \mathfrak{B} \beta e^{-\beta v} - \mathfrak{C} \gamma e^{-\gamma v}}{\mathfrak{A} e^{-\alpha v} + \mathfrak{B} e^{-\beta v} + \mathfrak{C} e^{-\gamma v}} \quad \text{et}$$

$$q = \frac{\mathfrak{A} \alpha \alpha e^{-\alpha v} + \mathfrak{B} \beta \beta e^{-\beta v} + \mathfrak{C} \gamma \gamma e^{-\gamma v}}{\mathfrak{A} e^{-\alpha v} + \mathfrak{B} e^{-\beta v} + \mathfrak{C} e^{-\gamma v}},$$

ob $q = pu = \frac{y\,dy}{x\,dx} = \frac{dy}{x\,dv}$, quod insigne est specimen integrationis methodo directa vix perficiendae.

Problema 115.

920. Sumto elemento dx constante, si ponatur haec aequatio $2y^3 ddy + yy\,dy^2 + X\,dx^2 = 0$, existente $X = \alpha + \beta x + \gamma xx$ inuenire multiplicatorem, qui eam integrabilem reddat.

Solutio.

Hic frustra tentantur multiplicatores formae

$$L\,dy + M\,dx \quad \text{et} \quad L\,dy^2 + M\,dx\,dy + N\,dx^2,$$

sumamus ergo multiplicatorem huius formae:

$$2L\,dy^3 + M\,dx^2 dy + N\,dx^3,$$

et

et integrale ſtatuatur

$$Ly^3 dy^4 + My^3 dx^2 dy^2 + 3Ny^3 dx^3 dy + S dx^4 = 0$$

vnde per differentiationem colligitur:

$$dx^4 dS = 2Lyy dy^5 + Myy dx^2 dy^3 + Nyy dx^3 dy^2 + MX dx^4 dy + NX dx^5$$
$$-3Lyy dy^5 + 2LX dx^2 dy^3 - y^3 dx^3 dy^2 \left(\frac{dM}{dx}\right) - 2y^3 dx^4 dy \left(\frac{dN}{dx}\right)$$
$$-y^3 dy^5 \left(\frac{dL}{dy}\right) - 3Myy dx^2 dy^3 - 6Nyy dx^3 dy^2$$
$$-y^3 dx^2 dy^3 \left(\frac{dM}{dy}\right) - 2y^3 dx^3 dy^2 \left(\frac{dN}{dy}\right)$$

vbi ſumimus L eſſe functionem ipſius y tantum Vt ergo termini dy^5 continentes deſtruantur, erit

$$-L - \frac{y\,dL}{dy} = 0 \text{ et } L = \frac{1}{y}.$$

Deinde pro deſtructione terminarum per dy^3 affectorum erit

$$-2Myy + \frac{2X}{y} - y^3\left(\frac{dM}{dy}\right) = 0,$$

et ſumto x conſtante

$$dM + \frac{2M\,dy}{y} = \frac{2X\,dy}{y^4},$$

quae per yy multiplicata et integrata praebet

$$Myy = P - \frac{2X}{y} \text{ et } M = \frac{P}{yy} - \frac{2X}{y^3}$$

denotante P functionem quamcunque ipſius x. Iam ad terminos dy^2 tollendos erit

$$-5Nyy - y\frac{dP}{dx} + \frac{2\,dX}{dx} - 2y^3\left(\frac{dN}{dy}\right) = 0$$

et ſumto x conſtante:

$$2y^3 dN + 5Nyy\,dy = \frac{2\,dX}{dx}dy - \frac{dP}{dx}.y\,dy$$

quae

quae per $\sqrt{y}$ diuisa et integrata dat:

$$2Ny^{\frac{3}{2}} = \frac{4dX}{dx}\sqrt{y} - \frac{2dP}{3dx}y\sqrt{y}$$

neglecta functione ipsius x addenda, quoniam irrationalitas $\sqrt{y}$ in calculum non ingreditur. Erit ergo $N = \frac{2dX}{yydx} - \frac{dP}{3ydx}$ ac propterea

$$dS = dy\left(\frac{PX}{yy} - \frac{2XX}{y^3} - \frac{4y\,ddX}{dx^2} + \frac{2yy\,ddP}{3dx^2}\right) + \frac{2XdX}{yy} - \frac{XdP}{3y}$$

vnde fit integrando

$$S = \frac{XX}{yy} - \frac{PX}{y} - \frac{2yy\,ddX}{dx^2} + \frac{2y^3\,ddP}{9dx^2}$$
$$+\int\left(\frac{PdX}{y} + \frac{2XdP}{3y} + \frac{2yy\,d^3X}{dx^2} - \frac{2y^3\,d^3P}{9dx^2}\right)$$

quae finite exprimetur si $P = 0$, cum ob $X = \alpha + \beta x + \gamma xx$ sit $d^3X = 0$. Quocirca habemus:

$$L = \frac{1}{y},\ M = -\frac{2X}{y^3} \text{ et } N = \frac{2dX}{yydx}$$

atque

$$S = \frac{XX}{yy} - \frac{2yy\,ddX}{dx^2} + \text{Const.}$$

vnde aequatio integralis est

$$y^2dy^4 - 2Xdx^2dy^2 + 4ydXdx^2dy + \frac{XXdx^4}{yy} - 2yydx^4ddX = Cdx^4$$

Aequatio ergo proposita

$$2y^3ddy + yydy^2 + dx^2(\alpha + \beta x + \gamma xx) = 0$$

integrabilis redditur multiplicata per:

$$\frac{2dy^3}{y} - \frac{2(\alpha + \beta x + \gamma xx)dx^2dy}{y^3} + \frac{2dx^3(\beta + 2\gamma x)}{yy}$$

tum vero eſt integrale:

$$y^2dy^4-2dx^2dy^2(\alpha+\beta x+\gamma xx)+4ydx^3dy(\beta 2\gamma x)$$
$$-4\gamma yydx^4+\frac{(\alpha+3x+\gamma xx)^2dx^4}{yy}=Cdx^4$$

ſeu

$$(yydy^2-(\alpha+\beta x+\gamma xx)dx^2)^2+4y^3dx^3dy(+\beta+2\gamma x)$$
$$-4\gamma y^4dx^4=Cyydx^4.$$

Scholion 1.

921. Integrale hoc ita eſt intricatum, vt alia methodo vix inueniri potuiſſe videatur, verum etiam ita eſt comparatum, vt nulla pateat methodus id porro integrandi, vnde prima integratio parum lucri attuliſſe eſt iudicanda. Quemadmodum autem in praecedente problemate integrale completum ex alio fonte hauſimus, ita hic ſimili modo integrale eruere licet, quod eo magis eſt notatu dignum, cum aequatio propoſita in ſe ſpectata ſolutu ſit difficillima. Ponamus ſcilicet itidem $dx=ydv$, et cum ſit

$$ddy=dx\,d.\frac{dy}{dx}=ydv\,d.\frac{dy}{ydv}$$

erit ſumendo iam elementum dv conſtans

$$ddy=ydv\left(\frac{ddy}{ydv}-\frac{dy^2}{yydv}\right)=ddy-\frac{dy^2}{y}.$$

Hinc noſtra aequatio induit hanc formam

$$2y^3ddy-yydy^2+yydv^2(\alpha+\beta x+\gamma xx)=0 \quad \text{ſeu}$$
$$2yddy-dy^2+dv^2(\alpha+\beta x+\gamma xx)=0$$

quae

quae denuo differentiata praebet:

$$2y\,d^3y + y\,dv^2(\beta + 2\gamma x) = 0 \quad \text{seu}$$

$$2\,d^3y + d\,v^2(\beta + 2\gamma x) = 0$$

differentietur iterum prodibitque

$$2\,d^4y + 2\gamma y\,dv^4 = 0 \quad \text{seu} \quad d^4y + \gamma y\,dv^4 = 0$$

quam aequationem si aliunde resoluere, valoremque ipsius y per v exprimere liceat, erit $x = \int y\,dv$ seu sine integratione $x = -\frac{d^3y}{\gamma\,dv^3} - \frac{\beta}{2\gamma}$. At manifestum est isti aequationi differentiali quarti ordinis satisfacere $y = e^{\lambda v}$ si fit $\lambda^4 + \gamma = 0$. Ponamus ergo $\gamma = -n^4$, et quatuor ipsius λ habebuntur valores $\lambda = \pm n$ et $\lambda = \pm n\sqrt{-1}$, vnde eius integrale completum est

$$y = \mathrm{A}\,e^{nv} + \mathrm{B}\,e^{-nv} + \mathrm{C}\,\text{fin.}(nv + \zeta)$$

hincque

$$x = + \frac{\mathrm{A}}{n}e^{nv} - \frac{\mathrm{B}}{n}e^{-nv} - \frac{\mathrm{C}}{n}\,\text{cof}(nv + \zeta) + \frac{\beta}{2n^4}$$

qui ergo valores quoque satisfacient aequationi inter x et y propositae, dummodo constantes A, B, C et ζ ita a se pendentes capiantur, vt quantitati quoque α conueniant. His nempe valoribus substitutis fieri debet $\alpha + \beta x - n^4 xx + \frac{2y\,ddy - dy^2}{dv^2} = 0$, vbi tantum terminos constantes considerasse sufficit, quibus accenseri debent ii, qui quadratum sinus cosinusue anguli $nv + \zeta$ continent, quippe ex quorum

 com-

combinatione quantitas conſtans exſurgit. Cum ergo ſit

$$2y = 2Ae^{nv} + 2Be^{-nv} + 2C\,\text{ſin.}(nv+\zeta)$$

$$\frac{ddy}{dv^2} = nnAe^{nv} + Bnne^{-nv} - nnC\,\text{ſin.}(nv+\zeta)$$

$$\frac{dy}{dv} = nAe^{nv} - nBe^{-nv} + nC\,\text{coſ.}(nv+\zeta)$$

$$x = \frac{A}{n}e^{nv} - \frac{B}{n}e^{-nv} - \frac{C}{n}\,\text{ſin.}(nv+\zeta) + \frac{\beta}{2n^4}$$

erit ſumtis terminis memoratis

$$\beta x = \frac{\beta\beta}{2n^4}$$

$$-n^4xx = 2ABnn - nnCC\,\text{coſ.}(nv+\zeta)^2 - \frac{\beta\beta}{4n^4}$$

$$\frac{2y\,ddy}{dv^2} = 4ABnn - 2nnCC\,\text{ſin.}(nv+\zeta)^2$$

$$-\frac{dy^2}{dv^2} = 2ABnn - nnCC\,\text{coſ.}(nv+\zeta)^2$$

ergo

$$\alpha + 8nnAB - 2nnCC + \frac{\beta\beta}{4n^4} = 0 \text{ ideoque}$$

$$C = \sqrt{\left(\frac{\alpha}{2nn} + \frac{\beta\beta}{8n^6} + 4AB\right)} \text{ vel}$$

$$\alpha = 2nn(CC - 4AB) - \frac{\beta\beta}{4n^4} \text{ et}$$

$$\alpha + \beta x + \gamma xx = 2nn(CC - AB) - \left(\frac{\beta}{2nn} - nnx\right)^2.$$

Manent ergo tres conſtantes A, B et ζ indeterminatae, ita vt nullum ſit dubium, quin formulae pro x et y datae integrale completum exhibeant.

Scho-

Scholion 2.

922. Aequationes differentio - differentiales, quas in his duobus problematibus tractauimus, ad similem formam reduci possunt. Prior enim

$$y(yddy + dy^2) + Xdx^2 = 0$$

existente $X = Ax$ vel $X = \alpha + \beta x$, si ponatur $ydy = \frac{1}{2}dz$ seu $yy = z$ induit hanc formam

$$\tfrac{1}{2}ddz\sqrt{z} + Xdx^2 = 0,$$

quae ope multiplicatoris $\frac{3\,dz^2}{4\sqrt{z}} + 3Xdx^2$ integrabilis redditur. Altera vero aequatio

$$yy(2yddy + dy^2) + Xdx^2 = 0$$

existente $X = \alpha + \beta x + \gamma xx$, posito $y = z^{\frac{2}{3}}$, fit $dy = \frac{2}{3}z^{-\frac{1}{3}}dz$ et $ddy = \frac{2}{3}z^{-\frac{1}{3}}ddz - \frac{2}{9}z^{-\frac{4}{3}}dz^2$ hinc $2yddy + dy^2 = \frac{4}{3}z^{-\frac{1}{3}}ddz$, sicque aequatio hanc induit formam $\frac{4}{3}z^{-\frac{5}{3}}ddz + Xdx^2 = 0$, quae integrabilis redditur ope huius multiplicatoris:

$$\frac{16dz^3}{27z^{\frac{5}{3}}} - \frac{4Xdx^2dz}{3z^{\frac{7}{3}}} + \frac{2dXdx^2}{z^{\frac{4}{3}}}.$$

Hinc colligimus pro aequatione $ddz + \frac{2Xdx^2}{\sqrt{z}} = 0$, fore multiplicatorem $dz^2 + 4Xdx^2\sqrt{z}$, pro aequatione autem $ddz + \frac{3Xdx^2}{4z\sqrt[3]{zz}} = 0$, multiplicatorem fore:

$$dz^3 - \frac{9Xdx^2dz}{4\sqrt[3]{zz}} + \tfrac{27}{8}dXdx^2\sqrt[3]{z}$$

seu sub vno conspectu:

pro aequatione	multiplicator erit
$ddz + \frac{Xdx^2}{\sqrt{z}} = 0$	$dz^2 + 2Xdx^2\sqrt{z}$
$ddz + \frac{Xdx^2}{z\sqrt[3]{zz}} = 0$	$dz^3 - \frac{3Xdx^2dz}{\sqrt[3]{zz}} + \frac{9}{2}dXdx^2\sqrt[3]{z}.$

Caeterum hae integrationes maxime sunt notatu dignae cum ex aequationibus differentialibus altioribus perfici queant. Ita cum ex hac aequatione vbi dv constans:

$$d^3y + Advddy + Bdv^2dy + Cydv^3 = 0 \text{ fit}$$

$$y = \mathfrak{A}e^{\alpha v} + \mathfrak{B}e^{\beta v} + \mathfrak{C}e^{\gamma v}$$

si fuerint α, β, γ, radices huius aequationis

$$r^3 + Ar^2 + Br + C = 0,$$

ponamus $dv = \frac{dx}{y}$, et cum sit

$$ddy = dvd.\frac{dy}{dv} = \frac{dx}{y}d.\frac{ydy}{dx} \text{ et}$$

$$d^3y = dv^2d.\frac{ddy}{dv^2} = dv^2d.\left(\frac{1}{dv}d.\frac{dy}{dv}\right) = \frac{dx^2}{yy}d.\left(\frac{y}{dx}d.\frac{ydy}{dx}\right)$$

si iam dx constans sumamus erit

$$ddy = ddy + \frac{dy^2}{y} \text{ et}$$

$$d^3y = \frac{1}{yy}d.y(yddy + dy^2) = d^3y + \frac{4dyddy}{y} + \frac{dy^3}{yy}$$

hincque per yy multiplicando:

$$yyd^3y + 4ydyddy + dy^3 + Adx(yddy + dy^2) + Bdx^2dy + Cdx^3 = 0$$

quae

quae integrata dat:

$$yyddy+ydy^2+Aydxdy+Bydx^2+(Cx+D)dx^2=0$$

quae ergo per superiora integrari potest.

Problema 116.

923. Definire conditiones functionum P, Q, R et L, M, N vt haec aequatio differentio-differentialis

$$ddy+Pdy^2+Qdxdy+Rdx^2=0$$

integrabilis reddatur multiplicatore

$$3Ldy^2+2Mdxdy+Ndx^2.$$

Solutio.

Facta multiplicatione integratio terminorum per ddy affectorum dat

$$Ldy^3+Mdxdy^2+Ndx^2dy,$$

quare ponatur integrale:

$$Ldy^3+Mdxdy^2+Ndx^2dy+Vdx^3=Cdx^3$$

cuius differentiale aequari debet formulae propositae in multiplicatorem ductae, vnde oritur:

$$\begin{array}{llllll} dx^3dV= & 3LP.dy^4 & +3LQ.dxdy^3 & +3LR.dx^2dy^2 & & \\ & & +2MP & +2MQ & +2MR\,dx^3dy & \\ & -\left(\frac{dL}{dy}\right) & -\left(\frac{dL}{dx}\right) & +NP & +NQ & +NR\,dx^4 \\ & & -\left(\frac{dM}{dy}\right) & -\left(\frac{dM}{dx}\right) & -\left(\frac{dN}{dx}\right) & \\ & & & -\left(\frac{dN}{dy}\right) & & \end{array}$$

Hic

Hic ergo fieri oportet:

$$3LP-\left(\frac{dL}{dy}\right)=0$$

$$3LQ+2MP-\left(\frac{dL}{dx}\right)-\left(\frac{dM}{dy}\right)=0$$

$$3LR+2MQ+NP-\left(\frac{dM}{dx}\right)-\left(\frac{dN}{dy}\right)=0.$$

Tum vero erit

$$dV=\left(2MR+NQ-\left(\frac{dN}{dx}\right)\right)dy+NRdx$$

quae formula integrabilis esse debet. Ex illis autem aequationibus colligitur:

$$P=\frac{1}{3L}\left(\frac{dL}{dy}\right);\ Q=\frac{1}{3L}\left(\frac{dL}{dx}\right)+\frac{1}{3L}\left(\frac{dM}{dy}\right)-\frac{2M}{9LL}\left(\frac{dL}{dy}\right)\ \text{et}$$

$$R=\frac{1}{3L}\left(\frac{dM}{dx}\right)+\frac{1}{3L}\left(\frac{dN}{dy}\right)-\frac{N}{9LL}\left(\frac{dL}{dy}\right)-\frac{2M}{9LL}\left(\frac{dL}{dx}\right)-\frac{2M}{9LL}\left(\frac{dM}{dy}\right)+\frac{4MM}{27L^3}\left(\frac{dL}{dy}\right).$$

Coroll. 1.

924. Si L, M et N fuerint functiones ipsius x tantum erit $P=0$, $Q=\frac{dL}{3Ldx}$ et $R=\frac{dM}{3Ldx}-\frac{2MdL}{9LLdx}$ hinc

$$dV=\left(\frac{2MdM}{3Ldx}-\frac{4MMdL}{9LLdx}+\frac{NdL}{3Ldx}-\frac{dN}{dx}\right)dy+\frac{NdM}{3L}-\frac{2MNdL}{9LL}$$

ac coefficiens ipsius dy debet esse constans. Quare per $L^{\frac{1}{3}}$ diuidendo habebitur:

$$\frac{Cdx}{\sqrt[3]{L}}=\frac{2MdM}{3L\sqrt[3]{L}}-\frac{4MMdL}{9LL\sqrt[3]{L}}+\frac{NdL}{3L\sqrt[3]{L}}-\frac{dN}{\sqrt[3]{L}}$$

et integrando

$$C\int\frac{dx}{\sqrt[3]{L}}=\frac{MM}{3L\sqrt[3]{L}}-\frac{N}{\sqrt[3]{L}}\ \text{seu}\ N=\frac{MM}{3L}-CL^{\frac{1}{3}}\int\frac{dx}{\sqrt[3]{L}}$$

ergo

ergo

$$V = Cy + \int \left(\frac{dM}{3L} - \frac{2MdL}{9LL} \right) \left(\frac{MM}{3L} - CL^{\frac{1}{3}} \int \frac{dx}{\sqrt[3]{L}} \right).$$

Coroll. 2.

925. Sit $M = S\sqrt[3]{L^2}$ erit

$$dM = dS\sqrt[3]{L^2} + \frac{2SdL}{3\sqrt[3]{L}} \text{ et}$$

$$V = Cy + \tfrac{1}{3}\int \frac{dS}{\sqrt[3]{L}} \left(\tfrac{1}{3}SS\sqrt[3]{L} - CL^{\frac{1}{3}} \int \frac{dx}{\sqrt[3]{L}} \right) \text{ seu}$$

$$V = Cy + \tfrac{1}{27}S^3 - \tfrac{1}{3}C\int dS \int \frac{dx}{\sqrt[3]{L}}$$

tum vero

$$N = \tfrac{1}{3}SS\sqrt[3]{L} - CL^{\frac{1}{3}} \int \frac{dx}{\sqrt[3]{L}} = \left(\tfrac{1}{3}SS - C\int \frac{dx}{\sqrt[3]{L}} \right) \sqrt[3]{L}$$

atque $P = 0$, $Q = \frac{dL}{3Ldx}$ et $R = \frac{dS}{3dx\sqrt[3]{L}}$. Quare haec aequatio

$$ddy + \frac{dLdy}{3L} + \frac{dSdx}{3\sqrt[3]{L}} = 0$$

integrabilis redditur multiplicatore:

$$3Ldy^2 + 2Sdxdy\sqrt[3]{L^2} + dx^2 \left(\tfrac{1}{3}SS - C\int \frac{dx}{\sqrt[3]{L}} \right) \sqrt[3]{L}$$

et integrale est:

$$Ldy^3 + Sdxdy^2\sqrt[3]{LL} + dx^2dy \left(\tfrac{1}{3}SS - C\int \frac{dx}{\sqrt[3]{L}} \right) \sqrt[3]{L} + Cydx^3$$
$$+ \tfrac{1}{27}S^3dx^3 - \tfrac{1}{3}Cdx^3\int dS \int \frac{dx}{\sqrt[3]{L}} = 0.$$

Coroll. 3.

926. Hic quidquid pro constante C assumatur, idem integrale prodire debet. Hinc si $C=0$, aequationis

$$ddy+\frac{dLdy}{2L}+\frac{dSdx}{3\sqrt[3]{L^2}}=0$$

multiplicator erit

$$3Ldy^2+2Sdxdy\sqrt[3]{L^2}+\tfrac{1}{3}SSdx^2\sqrt[3]{L}$$

et integrale

$$Ldy^3+Sdxdy^2\sqrt[3]{LL}+\tfrac{1}{3}SSdx^2dy\sqrt[3]{L}+\tfrac{1}{27}S^3dx^3=Ddx^3$$

seu $(dy\sqrt[3]{L}+\frac{1}{3}Sdx)^3=Ddx^3$.

Scholion 1.

927. Ex iisdem quoque conditionibus, si dentur functiones P, Q et R definiri poterunt functiones L, M, N, quatenus quidem postrema conditio integrabilitatis patitur. Veluti si sit $P=\frac{n}{y}$, $Q=0$ et R functio ipsius x tantum, puta $R=X$, vt habeatur haec aequatio:

$$ddy+\frac{ndy^2}{y}+Xdx^2=0$$

cuius multiplicator si sumatur

$$3Ldy^2+2Mdxdy+Ndx^2,$$

vt integrale sit

$$Ldy^3+Mdxdy^2+Ndx^2dy+Vdx^3=Cdx^3$$

erit

erit primo $\frac{3nL}{y} - \left(\frac{dL}{dy}\right) = 0$, et sumta x constante $\frac{dL}{L} = \frac{3n\,dy}{y}$, hinc $L = Sy^{3n}$, denotante S functionem ipsius x. Deinde est

$$\frac{2nM}{y} - y^{3n}\frac{dS}{dx} - \left(\frac{dM}{dy}\right) = 0$$

et sumta x constante

$$dM - \frac{2nM\,dy}{y} + \frac{dS}{dx}.y^{3n}\,dy = 0$$

quae per y^{-2n} multiplicata et integrata dat:

$$y^{-2n}M + \frac{dS}{(n+1)dx}y^{n+1} = T \text{ funct. ipsius } x.$$

Ergo

$$M = Ty^{2n} - \frac{dS}{(n+1)dx}y^{3n+1}.$$

Tertio fieri debet:

$$3SXy^{3n} + \frac{nN}{y} - \frac{dT}{dx}.y^{2n} + \frac{ddS}{(n+1)dx^2}.y^{3n+1} - \frac{dN}{dy} = 0$$

vnde sumta x constante:

$$dN - \frac{nN\,dy}{y} + \frac{dT}{dx}.y^{2n}dy - \frac{ddS}{(n+1)dx^2}y^{3n+1}dy - 3SXy^{3n}dy = 0$$

quae per y^{-n} multiplicata et integrata dat:

$$y^{-n}N + \frac{dT}{(n+1)dx}y^{n+1} - \frac{ddS}{2(n+1)^2dx^2}y^{2n+2} - \frac{3SX}{2n+1}y^{2n+1} = U f{:}\,x$$

seu $N = Uy^n - \frac{dT}{(n+1)dx}y^{2n+1} + \frac{ddS}{2(n+1)^2dx^2}y^{3n+2} + \frac{3SX}{2n+1}y^{3n+1}$.

Ex his autem fit:

$$dV = dy\left\{\begin{array}{l} 2TXy^{2n} - \frac{2XdS}{(n+1)dx}y^{3n+1} - \frac{dU}{dx}.y^n + \frac{ddT}{(n+1)dx^2}y^{2n+1} \\ - \frac{d^3S}{2(n+1)^2dx^3}y^{3n+2} - \frac{3d.SX}{(2n+1)dx}y^{3n+1} \end{array}\right\}$$

$$+ Xdx\left(Uy^n - \frac{dT}{(n+1)dx}y^{2n+1} + \frac{ddS}{2(n+1)^2dx^2}y^{3n+2} + \frac{3SX}{2n+1}y^{3n+1}\right)$$

 quae

quae formula vt integrationem admittat, esse oportet

$$2y^{2n}d.TX-2y^{2n+1}.\frac{d.XS}{(n+1)dx}-y^n.\frac{ddU}{dx}+y^{2n+1}.\frac{d^3T}{(n+1)dx^2}$$
$$-y^{2n+2}.\frac{d^4S}{2(n+1)^2dx^3}-3y^{2n+1}\frac{dd.SX}{(2n+1)dx}-nUXy^{n-1}dx+\frac{(2n+1)XdT}{(n+1)}y^{2n}$$
$$-\frac{(3n+2)XddS}{2(n+1)^2dx}y^{2n+1}-\frac{2(2n+1)SXXdx}{(2n+1)}y^{2n}=0$$

hic ergo singulae potestates ipsius y, quatenus sunt inaequales, seorsim destrui debent. Quare potestas y^{n-1} dat $U=0$; vnde etiam potestas y^n ad nihilum redigitur. Potestas y^{2n} dat

$$(2n+2)TdX+(2n+2)XdT+(2n+1)XdT=0$$

seu $X^{2n+2}T^{4n+3}=A$, at potestas y^{2n+1} praebet $d^3T=0$ seu $T=\alpha+\beta x+\gamma xx$. Potestas vero y^{2n} postulat $S=0$ nisi sit $n=-\frac{1}{2}$; quo casu etiam potestates y^{2n+1} et y^{2n+2} sponte euanescunt. Cum ergo sit $U=0$, $S=0$ et $T=\alpha+\beta x+\gamma xx$, hincque $X=B(\alpha+\beta x+\gamma xx)^{\frac{-4n-3}{2n+2}}$ haec aequatio

$$ddy+\frac{ndy^2}{y}+B(\alpha+\beta x+\gamma xx)^{\frac{-4n-3}{2n+2}}dx^2=0$$

integrabilis redditur ope multiplicatoris:

$$2(\alpha+\beta x+\gamma xx)y^{2n}dy-\frac{dx(\beta+2\gamma x)}{n+1}y^{2n+1}.$$

Scholion 2.

928. Quanquam plurimum abest, quominus haec methodus satis adhuc sit culta, tamen specimina in hoc capite tradita abunde declarant, quanta incre-

incrementa inde expectare queamus, vnde eius cultura maxime Geometris commendanda videtur. Quoniam igitur methodi, quibus in resolutione aequationum differentio-differentialium vti conuenit, satis luculenter sunt expositae, ad sequens caput progrediamur, vbi integrationem huiusmodi aequationum, quatenus quidem id commode fieri potest, per series infinitas ostendemus.

CAPVT VII.

DE RESOLVTIONE AEQVATIONIS $ddy + ax^n y dx^2 = 0$ PER SERIES INFINITAS.

Problema 117.

929.

Sumto elemento dx constante aequationem differentio-differentialem $ddy + ax^n y dx^2 = 0$ per seriem infinitam integrare.

Solutio.

Quaerimus hic seriem secundum potestates ipsius x progredientem, quae valorem ipsius y exprimat; et quia in altero aequationis nostrae termino quantitas x cum suo differentiali dx nullam, in altero vero $n+2$ dimensiones occupat, euidens est exponentes potestatum ipsius x differentia $n+2$ ascendere vel descendere debere.

I.

I. Ascendant primo exponentes, et fingatur series:

$$y = Ax^{\lambda} + Bx^{\lambda+n+2} + Cx^{\lambda+2n+4} + \text{etc. eritque}$$

$$\frac{ddy}{dx^2} = \lambda(\lambda-1)Ax^{\lambda-2} + (\lambda+n+2)(\lambda+n+1)Bx^{\lambda+n} + \text{etc.}$$

$$ax^n y = \quad . \quad . \quad . \quad . \quad . \quad . \quad aAx^{\lambda+n}$$

vnde patet primum terminum solitarium euanescere debere, vt sit $\lambda(\lambda-1)=0$. Quare capi oportet vel $\lambda=0$ vel $\lambda=1$, sicque duplex series obtinetur:

$$y = A + Bx^{n+2} + Cx^{2n+4} + Dx^{3n+6} + Ex^{4n+8} + \text{etc.}$$
$$+ \mathfrak{A}x + \mathfrak{B}x^{n+3} + \mathfrak{C}x^{2n+5} + \mathfrak{D}x^{3n+7} + \mathfrak{E}x^{4n+9} + \text{etc.}$$

substitutione ergo facta fieri oportet

$$0 = (n+2)(n+1)Bx^n + (2n+4)(2n+3)Cx^{2n+2} + (3n+6)(3n+5)Dx^{3n+4} \text{etc.}$$
$$+ aA \quad + \quad aB \quad + aC$$

$$0 = (n+3)(n+2)\mathfrak{B}x^{n+1} + (2n+5)(2n+4)\mathfrak{C}x^{2n+3} + (3n+7)(3n+6)\mathfrak{D}x^{3n+5} \text{etc.}$$
$$+ a\mathfrak{A} \quad + a\mathfrak{B} \quad + a\mathfrak{C}$$

vnde litteris A et $\mathfrak{A}$ arbitrio nostro relictis reliquae per eas ita determinantur:

$$B = \frac{-aA}{(n+1)(n+2)};\ C = \frac{-aB}{2(2n+3)(n+2)};\ D = \frac{-aC}{3(3n+5)(n+2)} \text{ etc.}$$

$$\mathfrak{B} = \frac{-a\mathfrak{A}}{(n+3)(n+2)};\ \mathfrak{C} = \frac{-a\mathfrak{B}}{2(2n+5)(n+2)};\ \mathfrak{D} = \frac{-a\mathfrak{C}}{3(3n+7)(n+2)} \text{ etc.}$$

sicque habebitur integrale completum ita expressum

$$y = A - \frac{aAx^{n+2}}{1(n+1).(n+2)} + \frac{a^2Ax^{2n+4}}{1.2(n+1)(2n+3).(n+2)^2} - \frac{a^3Ax^{3n+6}}{1.2.3(n+1)(2n+3)(3n+5).(n+2)^3} + \text{etc.}$$

$$+\mathfrak{A}x - \frac{a\mathfrak{A}x^{n+3}}{1(n+3).(n+2)} + \frac{a^2\mathfrak{A}x^{2n+5}}{1.2(n+3)(2n+5).(n+2)^2} - \frac{a^3\mathfrak{A}x^{3n+7}}{1.2.3(n+3)(2n+5)(3n+7).(n+2)^3} + \text{etc.}$$

II.

II. Descendant iam exponentes, et ficta serie

$$y = Ax^{\lambda} + Bx^{\lambda-n-2} + Cx^{\lambda-2n-4} + \text{etc.}$$

habebitur:

$$\frac{ddy}{dx^2} = \lambda(\lambda-1)Ax^{\lambda-2} + (\lambda-n-2)(\lambda-n-3)Bx^{\lambda-n-4} + \text{etc.}$$

$$ax^n y = aAx^{\lambda+n} + aBx^{\lambda-2} + \text{etc.}$$

vbi cum terminus $x^{\lambda+n}$ sui similem non habens, tolli nequit, ita vt hinc nulla aequationis resolutio obtineatur.

Coroll. 1.

930. Geminata series pro y inuenta, quoniam litterae A et $\mathfrak{A}$ arbitrio nostro relinquuntur, integrale completum aequationis differentio-differentialis $ddy + ax^n y dx^2 = 0$ exhibet; tribuendo autem litteris A et $\mathfrak{A}$ datos valores, integralia particularia nascentur.

Coroll. 2.

931. Si ponamus $n + 2 = m$, seu $n = m - 2$, huius aequationis $ddy + ax^{m-2} y dx^2 = 0$ integrale completum ita commodius exprimetur:

$$y = \begin{cases} A - \frac{aAx^m}{1(m-1).m} + \frac{a^2Ax^{2m}}{1.2(m-1)(2m-1).m^2} - \frac{a^3Ax^{3m}}{1.2.3(m-1)(2m-1)(3m-1).m^3} + \text{etc} \\ \mathfrak{A}x - \frac{a\mathfrak{A}x^{m+1}}{1(m+1).m} + \frac{a^2\mathfrak{A}x^{2m+1}}{1.2(m+1)(2m+1).m^2} - \frac{a^3\mathfrak{A}x^{3m+1}}{1.2.3(m+1)(2m+1)(3m+1).m^3} \end{cases}$$

Coroll. 3.

Coroll. 3.

932. Si exponens m fuerit positiuus et vnitate maior, hae series eo magis conuergunt, quo minor valor quantitati x tribuatur: aliis vero casibus in praxi hae series adhiberi nequeunt; nisi forte eae ipsae in alias conuergentes transformari possint.

Scholion. 1.

933. Dantur tamen casus, quibus hae series omni plane vsu destituuntur, quod euenit, si quispiam factorum denominatores constituentium euanescat, sicque omnes termini sequentes in infinitum excrescant, quibus casibus series in alias formas transmutari conuenit. Hic primo occurrit casus $m=0$ seu $n=-2$, quo vtriusque seriei omnes termini praeter primos fiunt infiniti, hoc vero casu aequatio, quae est $ddy+\frac{aydx^2}{x^2}=0$, cum sit homogenea singularem integrationem admittit: inueniri enim potest potestas ipsius x, quae pro y substituta aequationi satisfacit. Ponatur scilicet $y=x^\lambda$, prodibitque

$$\lambda(\lambda-1)x^{\lambda-2}+ax^{\lambda-2}=0 \text{ seu } \lambda\lambda-\lambda+a=0$$

vnde colligitur $\lambda=\frac{1}{2}\pm\sqrt{(\frac{1}{4}-a)}$, ob quem duplicem valorem est integrale completum:

$$y=Ax^{\frac{1}{2}+\sqrt{(\frac{1}{4}-a)}}+Bx^{\frac{1}{2}-\sqrt{(\frac{1}{4}-a)}}$$

 quae

quae casu $a > \frac{1}{4}$ abit in hanc formam

$$y = A x^{\frac{1}{2}} \text{sin.} ((a-\tfrac{1}{4})^{\frac{1}{2}} lx + \alpha)$$

vnde patet casu $a = \frac{1}{4}$ fore

$$y = (A + B lx) \sqrt{x}.$$

Scholion 2.

934. Reliqui casus ad incommodum ducentes sunt, si vel $m = \frac{1}{i}$ vel $m = -\frac{1}{i}$ denotante i numerum quemcunque integrum. Casu $m = \frac{1}{i}$ prior tantum series fit incongrua, casu vero $m = -\frac{1}{i}$ posterior tantum. Quare illo casu ponendo $A = 0$ hoc vero $\mathfrak{A} = 0$, series saltem vna idonea habetur integrale particulare exhibens. Verum cognito integrali particulari quod sit $y = P$, inde aequationis

$$ddy + a x^{m-2} y dx^2 = 0$$

integrale completum eruitur ponendo $y = Pz$ vnde fit

$$Pddz + 2dPdz + zddP + a x^{m-2} P z dx^2 = 0$$

at per hypothesin est

$$ddP + a x^{m-2} P dx^2 = 0,$$

ergo prodit

$$Pddz + 2dPdz = 0 \text{ seu } PPdz = Cdx \text{ et } z = C \int \frac{dx}{PP}.$$

Cum autem P sit series infinita, hinc valorem ipsius z cognoscere haud licet. At casibus illis memoratis pars integralis logarithmum ipsius x inuol-

vit,

vit, quod vel inde intelligitur quod $\frac{x^0}{0}$ aequiualeat ipsi lx. Quare in aequatione

$$ddy + ax^{m-2}ydx^2 = 0$$

ponendo $y = p + qlx$ ob $dy = dp + \frac{qdx}{x} + dqlx$ erit

$$ddp + \frac{2dxdq}{x} - \frac{qdx^2}{xx} + ddqlx + apx^{m-2}dx^2 + aqx^{m-2}dx^2lx = 0$$

in qua partes lx inuoluentes seorsim destruantur necesse est; ita vt hae binae habeantur aequationes:

$$ddq + aqx^{m-2}dx^2 = 0 \text{ et}$$

$$ddp + \frac{2dxdq}{x} - \frac{qdx^2}{xx} + apx^{m-2}dx^2 = 0$$

vbi pro q ea binarum superiorum serierum accipi debet, quae casu oblato incommodo caret, eaque constituta ex posteriori aequatione facile quantitas p per seriem exprimetur. Huiusmodi casus in sequentibus exemplis euoluamus; tantum notemus illam operationem perinde se habere, etiamsi loco lx sumatur $lx + \alpha$ ita vt inuentis p et q futurum sit

$$y = \alpha q + p + qlx \text{ seu } y = p + ql\beta x.$$

Exemplum 1.

935. *Posito* $m = 1$ *hanc aequationem*

$$ddy + \frac{aydx^2}{x} = 0$$

per series resoluere.

 Posito

Posito $y = p + q l x$, capi oportet:

$$q = \mathfrak{A}x - \frac{a\mathfrak{A}x^2}{1.2} + \frac{a^2\mathfrak{A}x^3}{1.2.2.3} - \frac{a^3\mathfrak{A}x^4}{1.2.2.3.3.4} + \text{etc.}$$

pro qua ponamus breuitatis gratia

$$q = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \text{etc.}$$

tum vero quaeratur p ex hac aequatione:

$$\frac{ddp}{dx^2} + \frac{2dq}{xdx} - \frac{q}{xx} + \frac{ap}{x} = 0$$

fingamus ergo

$$p = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 + \mathrm{E}x^4 + \text{etc.}$$

eritque facta substitutione:

$$\left.\begin{array}{llllll} & 2\mathrm{C} + 6\mathrm{D}x + 12\mathrm{E}xx + 20\mathrm{F}x^3 + 30\mathrm{G}x^4 & & & & \\ +\frac{2\mathfrak{A}}{x} & +4\mathfrak{B} & +6\mathfrak{C} & +8\mathfrak{D} & +10\mathfrak{E} & +12\mathfrak{F} \\ -\mathfrak{A} & -\mathfrak{B} & -\mathfrak{C} & -\mathfrak{D} & -\mathfrak{E} & -\mathfrak{F} \\ +a\mathrm{A} & +a\mathrm{B} & +a\mathrm{C} & +a\mathrm{D} & +a\mathrm{E} & +a\mathrm{F} \end{array}\right\} = 0.$$

Cum iam dentur coefficientes $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ etc. erit $\mathrm{A} = -\frac{\mathfrak{A}}{a}$, quantitas B non determinatur, tum vero

$$\mathrm{C} = \frac{-3\mathfrak{B}}{2} - \frac{a\mathrm{B}}{1.2} = \frac{+3a\mathfrak{A}}{1^2.2^2} - \frac{a\mathrm{B}}{1.2}$$

$$\mathrm{D} = \frac{-5\mathfrak{C}}{6} - \frac{a\mathrm{C}}{2.3} = \frac{-5a^2\mathfrak{A}}{1^2.2^3.3^2} - \frac{a\mathrm{C}}{2.3}$$

$$\mathrm{E} = \frac{-7\mathfrak{D}}{12} - \frac{a\mathrm{D}}{3.4} = \frac{+7a^3\mathfrak{A}}{1^2.2^2.3^3.4^2} - \frac{a\mathrm{D}}{3.4}$$

$$\mathrm{F} = \frac{-9\mathfrak{E}}{20} - \frac{a\mathrm{E}}{4.5} = \frac{-9a^4\mathfrak{A}}{1^2.2^2.3^2.4^3.5^2} - \frac{a\mathrm{E}}{4.5}$$

vbi pro B scribere licet o, quandoquidem in integrali $y = p + q l x$ addimus partem αq, quae ex littera

littera B oritur, ita vt sit

$$p = A + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

Hinc erit

$$C = \frac{3a\mathfrak{A}}{1^3 \cdot 2^2};\ D = \frac{-14a^2\mathfrak{A}}{1^3 \cdot 2^3 \cdot 3^2};\ E = \frac{+70a^3\mathfrak{A}}{1^3 \cdot 2^3 \cdot 3^3 \cdot 4^2};\ D = \frac{-404a^4\mathfrak{A}}{1^3 \cdot 2^3 \cdot 3^3 \cdot 4^3 \cdot 5^2}$$

vbi notetur esse:

$14 = 3.3 + 5.1;\ 70 = 4.14 + 7.1.2;\ 404 = 5.70 + 9.1.2.3$ et pro sequente

$2688 = 6.404 + 11.1\ 2.3.4$ estque $y = p + \alpha q + qlx$

Exemplum 2.

936. *Posito* $m = -1$ *hanc aequationem*

$$ddy + \frac{aydx^2}{x^3} = 0$$

per series resoluere.

Posito $y = p + \alpha q + qlx$ capi oportet:

$$q = A - \frac{aA}{1 \cdot 2x} + \frac{a^2A}{1 \cdot 2^2 \cdot 3x^2} - \frac{a^3A}{1 \cdot 2^2 \cdot 3^2 \cdot 4x^3} + \text{etc.}$$

pro qua ponatur breuitatis gratia:

$$q = A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3} + \frac{E}{x^4} + \text{etc.}$$

tum vero quantitas p ex hac aequatione definiri debet:

$$ddp + \frac{2dqdx}{x} - \frac{qdx^2}{xx} + \frac{apdx^2}{x^3} = 0.$$

Fingamus ergo

$$p = \mathfrak{A}x + \mathfrak{B} + \frac{\mathfrak{C}}{x} + \frac{\mathfrak{D}}{xx} + \frac{\mathfrak{E}}{x^3} + \frac{\mathfrak{F}}{x^4} + \text{etc.}$$

vnde facta ſubſtitutione prodit

$$\left.\begin{array}{l} \frac{a\mathfrak{A}}{xx}+\frac{a\mathfrak{B}}{x^3}+\frac{a\mathfrak{C}}{x^4}+\frac{a\mathfrak{D}}{x^5}+\frac{a\mathfrak{E}}{x^6}+\frac{a\mathfrak{F}}{x^7}+\text{etc.} \\ -\frac{A}{xx}-\ B-\ C-\ D-\ E-\ F \\ \quad -2B-4C-6D-8E-10F \\ \quad +2\mathfrak{C}+6\mathfrak{D}+12\mathfrak{E}+20\mathfrak{F}+30\mathfrak{G} \end{array}\right\}=0$$

et coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ etc. ita determinantur, vt ſit $\mathfrak{A}=\frac{A}{a}$; ſecundus $\mathfrak{B}$ non definitur, tum vero eſt

$$\mathfrak{C}=\frac{3B}{1\cdot 2}-\frac{a\mathfrak{B}}{1\cdot 2}=\frac{-3aA}{1^2\cdot 2^2}\quad-\frac{a\mathfrak{B}}{1\cdot 2}$$

$$\mathfrak{D}=\frac{5C}{2\cdot 3}-\frac{a\mathfrak{C}}{2\cdot 3}=\frac{5a^2A}{2^2\cdot 3^2}\quad-\frac{a\mathfrak{C}}{2\cdot 3}$$

$$\mathfrak{E}=\frac{7D}{3\cdot 4}-\frac{a\mathfrak{D}}{3\cdot 4}=\frac{7a^3A}{2^2\cdot 3^2\cdot 4^2}\quad-\frac{a\mathfrak{D}}{3\cdot 4}$$

$$\mathfrak{F}=\frac{9E}{4\cdot 5}-\frac{a\mathfrak{E}}{4\cdot 5}=\frac{9a^4A}{2^2\cdot 3^2\cdot 4^2\cdot 5^2}-\frac{a\mathfrak{E}}{4\cdot 5}$$

ſi ſumatur, id quod ſine detrimento generalitatis fieri licet, $\mathfrak{B}=0$, ita vt ſit

$$p=\mathfrak{A}x+\frac{\mathfrak{C}}{x}+\frac{\mathfrak{D}}{xx}+\frac{\mathfrak{E}}{x^3}+\frac{\mathfrak{F}}{x^4}+\text{etc., erit}$$

$$\mathfrak{C}=\frac{-3aA}{1^2\cdot 2^2}\ ;\ \mathfrak{D}=\frac{+14a^2A}{1^2\cdot 2^2\cdot 3^2}\ ;\ \mathfrak{E}=\frac{-90a^3A}{1^2\cdot 2^2\cdot 3^2\cdot 4^2}\ ;$$

$$\mathfrak{F}=\frac{+404a^4A}{1^2\cdot 2^2\cdot 3^2\cdot 4^2\cdot 5^2}\ \text{etc.}$$

qui valores ſimiles ſunt praecedentibus.

Exem-

Exemplum 3.

937. *Posito* $m = \frac{1}{2}$ *hanc aequationem*

$$ddy + \frac{aydx^2}{x\sqrt{x}} = 0$$

per series resoluere.

Posito $y = p + \alpha q + qlx$ capi oportet:

$$q = \mathfrak{A}x - \frac{4a\mathfrak{A}}{1.3}x^{\frac{3}{2}} + \frac{16a^2\mathfrak{A}}{1.2.3.4}x^2 - \frac{64a^3\mathfrak{A}}{1.2.3.3.4.5}x^{\frac{5}{2}} + \frac{256a^4\mathfrak{A}}{1.2.3.4.3.4.5.6}x^3 \text{ etc.}$$

pro qua breuitatis gratia scribatur

$$q = \mathfrak{A}x + \mathfrak{B}x^{\frac{3}{2}} + \mathfrak{C}x^2 + \mathfrak{D}x^{\frac{5}{2}} + \mathfrak{E}x^3 + \mathfrak{F}x^{\frac{7}{2}} + \mathfrak{G}x^4 + \text{etc.}$$

tum vero quantitas p ex hac aequatione definiri debet:

$$ddp + \frac{2dxdq}{x} - \frac{qdx^2}{xx} + \frac{apdx^2}{x\sqrt{x}} = 0.$$

Fingamus ergo

$$p = \Delta + Ax^{\frac{1}{2}} + Bx + Cx^{\frac{3}{2}} + Dx^2 + Ex^{\frac{5}{2}} + Fx^3 + \text{etc.}$$

prodibitque facta substitutione

$$\left.\begin{array}{llllllll} \frac{a\Delta}{x\sqrt{x}} + & \frac{aA}{x} + & \frac{aB}{\sqrt{x}} & + aC & + aD\sqrt{x} & + aEx & + aFx\sqrt{x} \\ & - \mathfrak{A} & - \mathfrak{B} & - \mathfrak{C} & - \mathfrak{D} & - \mathfrak{E} & - \mathfrak{F} \\ & + 2\mathfrak{A} & + 3\mathfrak{B} & + 4\mathfrak{C} & + 5\mathfrak{D} & + 6\mathfrak{E} & + 7\mathfrak{F} \\ - \frac{A}{4} & + 0 & + \frac{3}{4}C & + 2D & + \frac{15}{4}E & + 6F & + \frac{35}{4}G \end{array}\right\} = 0$$

Hinc

Hinc colligitur fore:

$$A = \frac{-\mathfrak{A}}{a}; \quad \Delta = \frac{-\mathfrak{A}}{4aa},$$

et B non determinatur, porro

$$C = \frac{-4aB}{1 \cdot 3} - \frac{8\mathfrak{B}}{1 \cdot 3} = \frac{-4aB}{1 \cdot 3} + \frac{32a\mathfrak{A}}{1^2 \cdot 3^2}$$

$$D = \frac{-4aC}{2 \cdot 4} - \frac{12\mathfrak{C}}{2 \cdot 4} = \frac{-4aC}{2 \cdot 4} - \frac{12 \cdot 16a^2\mathfrak{A}}{1 \cdot 3 \cdot 2^2 \cdot 4^2}$$

$$E = \frac{-4aD}{3 \cdot 5} - \frac{16\mathfrak{D}}{3 \cdot 5} = \frac{-4aD}{3 \cdot 5} + \frac{16 \cdot 64a^3\mathfrak{A}}{1 \cdot 3 \cdot 2 \cdot 4 \cdot 3^2 \cdot 5^2}$$

$$F = \frac{-4aE}{4 \cdot 6} - \frac{20\mathfrak{E}}{4 \cdot 6} = \frac{-4aE}{4 \cdot 6} - \frac{20 \cdot 256a^4\mathfrak{A}}{1 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot 5 \cdot 4^2 \cdot 6^2}.$$

Quodſi iam ponatur $B = 0$ vt ſit

$$p = \frac{-\mathfrak{A}}{4aa} - \frac{\mathfrak{A}}{a}\sqrt{x} + * + Cx\sqrt{x} + Dx^2 + Ex^2\sqrt{x} + Fx^3 + \text{etc.}$$

erit

$$C = \frac{8 \cdot 4a\mathfrak{A}}{1^2 \cdot 3^2}; \quad D = \frac{-100 \cdot 16a^2\mathfrak{A}}{1^2 \cdot 3^2 \cdot 2^2 \cdot 4^2}; \quad E = \frac{1884 \cdot 64a^3\mathfrak{A}}{1^2 \cdot 3^2 \cdot 2^2 \cdot 4^2 \cdot 3^2 \cdot 5^2};$$

$$F = \frac{-52416 \cdot 256a^4\mathfrak{A}}{1^2 \cdot 3^2 \cdot 2^2 \cdot 4^2 \cdot 3^2 \cdot 5^2 \cdot 4^2 \cdot 6^2} \text{ etc.}$$

vbi notetur eſſe:

$100 = 2.4.8 + 1.3.12$; $1884 = 3.5.100 + 1.3.2.4.16$;
$52416 = 4.6.1884 + 1.3.2.4.3.5.20$.

Exemplum 4.

938. *Poſito* $m = -\frac{1}{2}$ *hanc aequationem*

$$ddy + \frac{aydx^2}{xx\sqrt{x}} = 0$$

per ſeries reſoluere.

Poſito $y = p + \alpha q + qlx$ capi oportet:

$$q = A - \frac{4aA}{1 \cdot 3}x^{-\frac{1}{2}} + \frac{16a^2A}{1 \cdot 3 \cdot 2 \cdot 4}x^{-1} - \frac{64a^3A}{1 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot 5}x^{-\frac{3}{2}} + \text{etc.}$$

pro

pro qua breuitatis cauſa ſcribamus

$$q = A + Bx^{-\frac{1}{2}} + Cx^{-1} + Dx^{-\frac{3}{2}} + Ex^{-2} + Fx^{-\frac{5}{2}} + \text{etc.}$$

et littera p ex hac aequatione difiniri debet

$$ddp + \frac{2dxdq}{x} - \frac{qdx^2}{xx} + \frac{apdx^2}{xx\sqrt{x}} = 0.$$

Fingamus

$$p = \Delta x + \mathfrak{A}\sqrt{x} + \mathfrak{B} + \mathfrak{C}x^{-\frac{1}{2}} + \mathfrak{D}x^{-1} + \mathfrak{E}x^{-\frac{3}{2}} + \mathfrak{F}x^{-2} + \mathfrak{G}x^{-\frac{5}{2}} + \text{etc.}$$

et facta ſubſtitutione prodit

$$\left.\begin{array}{ccccccc} \frac{a\Delta}{x\sqrt{x}} & + \frac{a\mathfrak{A}}{xx} & + \frac{a\mathfrak{B}}{xx\sqrt{x}} & + \frac{a\mathfrak{C}}{x^3} & + \frac{a\mathfrak{D}}{x^3\sqrt{x}} & + \frac{a\mathfrak{E}}{x^4} & + \frac{a\mathfrak{F}}{x^4\sqrt{x}} \\ & -A & -B & -C & -D & -E & -F \\ & & -B & -2C & -3D & -4E & -5F \\ -\frac{1}{4}\mathfrak{A} & & +\frac{3}{4}\mathfrak{C} & +2\mathfrak{D} & +\frac{15}{4}\mathfrak{E} & +6\mathfrak{F} & +\frac{35}{4}\mathfrak{G} \end{array}\right\} = 0$$

vnde ſequentes determinationes colliguntur:

$$\mathfrak{A} = \frac{A}{a} \text{ et } \Delta = \frac{\mathfrak{A}}{4a} = \frac{A}{4aa},$$

at $\mathfrak{B}$ non determinatur: porro

$$\mathfrak{C} = \frac{-4a\mathfrak{B}}{1\cdot 3} + \frac{8B}{1\cdot 3} = \frac{-4a\mathfrak{B}}{1\cdot 3} - \frac{8\cdot 4aA}{1^2\cdot 3^2}$$

$$\mathfrak{D} = \frac{-4a\mathfrak{C}}{2\cdot 4} + \frac{12C}{2\cdot 4} = \frac{-4a\mathfrak{C}}{2\cdot 4} + \frac{12\cdot 16a^2A}{1\cdot 3\cdot 2^2\cdot 4^2}$$

$$\mathfrak{E} = \frac{-4a\mathfrak{D}}{3\cdot 5} + \frac{16D}{3\cdot 5} = \frac{-4a\mathfrak{D}}{3\cdot 5} - \frac{16\cdot 64a^3A}{1\cdot 3\cdot 2\cdot 4\cdot 3^2\cdot 5^2}$$

$$\mathfrak{F} = \frac{-4a\mathfrak{E}}{4\cdot 6} + \frac{20E}{4\cdot 6} = \frac{-4a\mathfrak{E}}{4\cdot 6} + \frac{20\cdot 256a^4A}{1\cdot 3\cdot 2\cdot 4\cdot 3\cdot 5\cdot 4^2\cdot 6^2}.$$

Quodsi iam sumatur $\mathfrak{B}=0$, erit

$\mathfrak{C}=\frac{-8.4aA}{1^2.3^2}$; $\mathfrak{D}=\frac{+100.16a^2A}{1^2.5^2.2^2.4^2}$; $\mathfrak{C}=\frac{-11664.64a^3A}{1^2.3^2.2^2.4^2.8^2.5^2}$ etc

qui numeri vt ante progrediuntur.

Scholion.

939. Ex his exemplis perspicitur quomodo series aequationem

$$ddy+ax^{m-2}ydx^2=0$$

resoluentes in reliquis casibus, quibus $m=\pm\frac{k}{i}$ inveniri oporteat; vbi obseruetur, si sit $m=+\frac{k}{i}$ pro q hanc seriem accipi debere:

$$q=\mathfrak{A}x+\mathfrak{B}x^{1+\frac{k}{i}}+\mathfrak{C}x^{1+\frac{2k}{i}}+\mathfrak{D}x^{1+\frac{3k}{i}}+\text{etc.}$$

tum vero formam ipsius p tali serie exprimi:

$$p=A+Bx^{\frac{k}{i}}+Cx^{\frac{2k}{i}}+Dx^{\frac{3k}{i}}+\text{etc.}$$

cuius coefficientes ex superioribus vt ante definiantur. Sin autem sit $m=-\frac{k}{i}$, pro q sumatur series:

$$q=A+Bx^{-\frac{k}{i}}+Cx^{-\frac{2k}{i}}+Dx^{-\frac{3k}{i}}+\text{etc.}$$

at pro p huiusmodi formam accipi conueniet:

$$p=\mathfrak{A}x+\mathfrak{B}x^{1-\frac{k}{i}}+\mathfrak{C}x^{1-\frac{2k}{i}}+\mathfrak{D}x^{1-\frac{3k}{i}}+\text{etc.}$$

vnde pariter singulos coefficientes vno excepto determinare licebit. Atque hoc artificium in genere est ten-

tenendum, quoties in resolutione aequationis generalis ad series peruenitur, cuius coefficientes certis casibus in infinitum excrescunt, quod plerumque indicio est logarithmos esse introducendos. Verum etiam eadem aequatio $ddy + ax^n ydx^2 = 0$ aliis modis per series resolui potest, dum ea ante resolutionem in aliam formam transmutatur, vbi cum euenire possit, vt series certis casibus abrumpatur, quibus adeo integrale reuera assignari potest, talem transformationem maxime notabilem hic explicemus.

Problema 118.

940. Aequationem differentio-differentialem $ddy + ax^n ydx^2 = 0$ in aliam formam transfundere, cuius resolutio per series infinitas commode institui possit.

Solutio.

Vtamur substitutione $y = e^{\int pdx} z$, vbi p sit certa functio ipsius x aequationem commode resolubilem suppeditans. Erit ergo

$$dy = e^{\int pdx}(dz + pzdx) \text{ et}$$

$$ddy = e^{\int pdx}(ddz + 2pdxdz + zdxdp + ppzdx^2),$$

vnde aequatio proposita abit in

$$ddz + 2pdxdz + zdxdp + ppzdx^2 + ax^n zdx^2 = 0$$

vbi p ita capiatur, vt fiat

$$pp + ax^n = 0 \text{ seu } p = x^{\frac{n}{2}}\sqrt{-a}.$$

Ponamus ideo $a = -cc$ et $n = 2m$, vt propofita fit haec aequatio

$$ddy - ccx^{2m}ydx^2 = 0,$$

quae pofito

$$p = cx^m \text{ et } y = e^{\int pdx}z = e^{\frac{c}{m+1}x^{m+1}}z,$$

induct hanc formam:

$$ddz + 2cx^m dxdz + mcx^{m-1}zdx^2 = 0$$

in qua cum x occupet vel nullam, vel $m+1$ dimenfiones, fingamus ipfius z valorem:

$$z = Ax^\lambda + Bx^{\lambda+m+1} + Cx^{\lambda+2m+2} + \text{etc.}$$

quo fubftituto fit

$$\begin{aligned}\lambda(\lambda-1)Ax^{\lambda-2} + (\lambda+m+1)(\lambda+m)&Bx^{\lambda+m-1} = 0\\ +2\lambda A&c\\ +mA&c\end{aligned}$$

vnde perfpicuum eft fumi debere vel $\lambda = 0$ vel $\lambda = 1$. Confequimur ergo feriem duplicatam huiusmodi:

$$\begin{aligned}z = A + Bx^{m+1} + Cx^{2m+2} + Dx^{3m+3} + Ex^{4m+4} + \text{etc.}\\ \mathfrak{A}x + \mathfrak{B}x^{m+2} + \mathfrak{C}x^{2m+3} + \mathfrak{D}x^{3m+4} + \mathfrak{E}x^{4m+5} + \text{etc.}\end{aligned}$$

qua

qua ſubſtituta fit

$$\left.\begin{array}{lll}(m+1)m\mathrm{B}x^{m-1} & +2(m+1)(2m+1)\mathrm{C}x^{2m} & +3(m+1)(3m+2)\mathrm{D}x^{3m+1} \\ +m\mathrm{A}c & +2(m+1)\mathrm{B}c & +4(m+1)\mathrm{C}c \\ & +m\mathrm{B}c & +m\mathrm{C}c\end{array}\right\}=0$$

$$\left.\begin{array}{lll}+(m+1)(m+2)\mathfrak{B}x^{m} & +2(m+1)(2m+3)\mathfrak{C}x^{2m+1} & +3(m+1)(3m+4)\mathfrak{D}x^{3m+2} \\ +2\mathfrak{A}c & +2(m+2)\mathfrak{B}c & +2(2m+3)\mathfrak{C}c \\ +m\mathfrak{A}c & +m\mathfrak{B}c & +m\mathfrak{C}c\end{array}\right\}=0$$

vnde vtrique coefficientes ſequenti modo determinantur:

$$\begin{array}{l|l} \mathrm{B}=\frac{-m\mathrm{A}c}{m(m+1)} & \mathfrak{B}=\frac{-(m+2)\mathfrak{A}c}{(m+2)(m+1)} \\ \mathrm{C}=\frac{-(3m+2)\mathrm{B}c}{2(2m+1)(m+1)} & \mathfrak{C}=\frac{-(3m+4)\mathfrak{B}c}{2(2m+3)(m+1)} \\ \mathrm{D}=\frac{-(5m+4)\mathrm{C}c}{3(3m+2)(m+1)} & \mathfrak{D}=\frac{-(5m+6)\mathfrak{C}c}{3(3m+4)(m+1)} \\ \mathrm{E}=\frac{-(7m+6)\mathrm{D}c}{4(4m+3)(m+1)} & \mathfrak{E}=\frac{-(7m+8)\mathfrak{D}c}{4(4m+5)(m+1)} \\ \text{etc.} & \text{etc.} \end{array}$$

vbi bini coefficientes A et $\mathfrak{A}$ manent indeterminati, ita vt hoc integrale completum ſit cenſendum.

Aliter. Sumta ſerie, in qua exponentes, ipſius x decreſcant, fieri debet

$$2\lambda+m=0 \text{ ſeu } m=-2\lambda,$$

vt aequatio noſtra ſit

$$ddy-ccx^{-4\lambda}ydx^2=0,$$

quae poſito

$$p=cx^{-2\lambda} \text{ et } y=e^{\int pdx}z=e^{\frac{-c}{2\lambda-1}x^{-2\lambda+1}}z,$$

 abit

abit in

$$ddz + 2cx^{-2\lambda}dxdz - 2\lambda cx^{-2\lambda-1}zdx^2 = 0.$$

Ponamus ergo

$$z = Ax^{\lambda} + Bx^{3\lambda-1} + Cx^{5\lambda-2} + Dx^{7\lambda-3} + \text{etc.}$$

et substitutione facta prodit:

$$\left.\begin{array}{l} \lambda(\lambda-1)Ax^{\lambda-2} + (3\lambda-1)(3\lambda-2)Bx^{3\lambda-3} + (5\lambda-2)(5\lambda-3)Cx^{5\lambda-4} \\ +2\lambda Acx^{-\lambda-1} + 2(3\lambda-1)Bc + 2(5\lambda-2)Cc \qquad + 2(7\lambda-3)Dc \\ -2\lambda Ac \qquad -2\lambda Bc \qquad -2\lambda Cc \qquad\qquad -2\lambda Dc \end{array}\right\} = 0$$

vnde coefficientes ita determinantur:

$$B = \frac{-\lambda(\lambda-1)A}{(4\lambda-2)c} = \frac{-\lambda(\lambda-1)A}{2(2\lambda-1)c}$$

$$C = \frac{-(3\lambda-1)(3\lambda-2)B}{(8\lambda-4)c} = \frac{-(3\lambda-1)(3\lambda-2)B}{4(2\lambda-1)c}$$

$$D = \frac{-(5\lambda-4)(5\lambda-3)C}{(12\lambda-6)c} = \frac{-(5\lambda-2)(7\lambda-3)C}{6(2\lambda-1)c}$$

$$E = \frac{-(7\lambda-3)(7\lambda-4)D}{(16\lambda-8)c} = \frac{-(7\lambda-3)(7\lambda-4)D}{8(2\lambda-1)c}$$

etc.

Hic vnius tantum litterae A valor arbitrio nostro relinquitur, ex quo haec series tantum integrale particulare exhibet.

Coroll. 1.

941. Ex solutione priori patet, alteram seriem terminari quoties

$$(2i+1)m + 2i = 0 \text{ seu } m = \frac{-2i}{2i+1},$$

alteram

alteram vero quoties

$$(2i-1)m+2i=0 \text{ ſeu } m=\frac{-2i}{2i-1},$$

denotante i numerum integrum quemcunque. His ergo caſibus integrale ſaltem particulare finite exprimi poteſt.

Coroll. 2.

942. Altera ſolutio praebet ſeriem finitam, quoties fuerit vel

$$(2i+1)\lambda-i=0 \text{ vel } (2i-1)\lambda-i=0,$$

hoc eſt

$$\lambda=\frac{i}{2i\pm 1} \text{ et } m=\frac{-2i}{2i\pm 1},$$

vt ante. Reliquis vero caſibus haec ſeries in infinitum excurrit.

Coroll. 3.

943. Caſus ergo, quibus haec aequatio

$$ddy-ccx^n dx^2=0$$

atque adeo poſito $y=e^{\int u dx}$ etiam haec

$$du+uudx=ccx^n dx$$

integrationem ſaltem particularem admittit, ſunt $n=\frac{-4i}{2i\pm 1}$ ſumendo pro i numerum integrum quemcunque.

Scholion.

944. Sufficit autem integrale particulare inveniſſe, cum ex eo facile integrale completum erui poſſit.

possit. Cum enim in integrali insit littera c, dum aequatio differentialis tantum quadratum cc continet, perinde est siue in integrali sumatur $+c$ siue $-c$. Hinc si integrale particulare sit $y=P+cQ$, erit etiam $y=P-cQ$ integrale particulare, vnde integrale completum erit

$$y=\alpha(P+cQ)+\beta(P-cQ) \text{ seu } y=\alpha P+\beta cQ.$$

Quo haec clarius explicentur, ad solutionem alteram aequationis $ddy-ccx^{-4\lambda}ydx^2=0$ accommodentur, pro qua ponendo breuitatis gratia $\frac{1}{1-2\lambda}x^{1-2\lambda}=t$, fecimus $y=e^{ct}z$, et inuenimus:

$$z=Ax^{\lambda}-\frac{\lambda(\lambda-1)A}{2(2\lambda-1)c}x^{3\lambda-1}+\frac{\lambda(\lambda-1)(3\lambda-1)(3\lambda-2)A}{2\cdot 4(2\lambda-1)^2cc}x^{5\lambda-2}$$
$$-\frac{\lambda(\lambda-1)(3\lambda-1)(3\lambda-2)(5\lambda-2)(5\lambda-3)A}{2\cdot 4\cdot 6(2\lambda-1)^3c^3}x^{7\lambda-3}+\text{etc.}$$

Pro qua expressione distinguendo terminos per potestates pares ipsius c diuisos ab iis, qui per potestates impares sunt diuisi, scribamus $z=P-cQ$, ita vt iam P et Q tantum potestates pares ipsius c contineant eritque integrale particulare vnum

$$y=e^{ct}(P-cQ)$$

et alterum

$$y=e^{-ct}(P+cQ),$$

vnde completum erit

$$y=\tfrac{1}{2}P(\alpha e^{ct}+\beta e^{-ct})-\tfrac{1}{2}cQ(\alpha e^{ct}-\beta e^{-ct}).$$

Hinc

Hinc si c sit numerus imaginarius seu $cc = -bb$ vt aequatio sit

$$ddy + bbx^{-4\lambda} y dx^2 = 0, \text{ erit}$$

$z = P - bQ\sqrt{-1}$ et $e^{ct} = e^{bt\sqrt{-1}} = \cos. bt + \sqrt{-1}.\sin. bt$

ergo $y = P\left(\frac{\alpha+\beta}{2}\cos. bt + \frac{\alpha-\beta}{2}\sqrt{-1}.\sin. bt\right)$

$$-bQ\left(\frac{\alpha-\beta}{2}\cos. bt + \frac{\alpha+\beta}{2}\sqrt{-1}.\sin. bt\right)\sqrt{-1}$$

sit $\frac{\alpha+\beta}{2} = \gamma$ et $\frac{\alpha-\beta}{2}\sqrt{-1} = \delta$,

atque integrale completum hoc casu ita exprimetur:

$y = P(\gamma\cos. bt + \delta\sin. bt) - bQ(\delta\cos. bt - \gamma\sin. bt)$ seu

$y = (\gamma P - \delta bQ)\cos. bt + (\delta P + \gamma bQ)\sin. bt.$

Casus ergo hoc modo integrabiles euoluamus.

Exemplum 1.

945. *Integrale aequationis* $ddy - ccydx^2 = 0$ *inuenire.*

Hic est $\lambda = 0$ et $z = A$; atque $t = x$, vnde ob $P = A$ et $Q = 0$ erit integrale completum

$$y = \alpha e^{cx} + \beta e^{-cx}.$$

Casu autem $cc = -bb$ aequationis $ddy + bbydx^2 = 0$ integrale completum erit

$$y = \gamma\cos. bx + \delta\sin. bx.$$

Exemplum 2.

946. *Integrale aequationis* $ddy - ccx^{-4}ydx^2 = 0$ *inuenire.*

Hic ob $\lambda = 1$ eſt $z = Ax$, et $t = -\frac{1}{x}$, vnde ob $P = x$ et $Q = 0$, fit

$$y = (\alpha e^{ct} + \beta e^{-ct})x.$$

Caſu autem $cc = -bb$ aequationis

$$ddy + bbx^{-4}ydx^2 = 0$$

integrale eſt

$$y = (\alpha \operatorname{cof}.bt + \beta \operatorname{fin}.bt)x \text{ exiſtente } t = -\frac{1}{x}.$$

Exemplum 3.

947. *Integrale aequationis* $ddy - ccx^{-\frac{4}{3}}ydx^2 = 0$ *inuenire.*

Ob $\lambda = \frac{1}{3}$, fit $B = -\frac{A}{3c}$, et $z = Ax^{\frac{2}{3}} - \frac{A}{3c}$, atque $t = 3x^{\frac{1}{3}}$ vnde $P = x^{\frac{2}{3}}$ et $Q = \frac{1}{3cc}$. Integrale ergo erit:

$$y = (\alpha e^{ct} + \beta e^{-ct})x^{\frac{2}{3}} - (\alpha e^{ct} - \beta e^{-ct})\frac{1}{3.c}.$$

Caſu autem $cc = -bb$ aequationis

$$ddy + bbx^{-\frac{4}{3}}ydx^2 = 0$$

integrale eſt

$$y = (\alpha \operatorname{cof}.bt + \beta \operatorname{fin}.bt)x^{\frac{2}{3}} + \frac{1}{3.b}(\beta \operatorname{cof}.bt - \alpha \operatorname{fin}.bt).$$

Exem-

Exemplum 4.

948. *Integrale aequationis* $ddy - ccx^{-\frac{8}{3}} y dx^2 = 0$ *inuenire.*

Ob $\lambda = \frac{2}{3}$ fit $B = \frac{A}{3c}$ et $z = Ax^{\frac{2}{3}} + \frac{A}{3c}x$, ita vt fit $P = x^{\frac{2}{3}}$ et $Q = \frac{-x}{3cc}$. Pofito ergo $t = -3x^{-\frac{1}{3}}$, integrale ita exprimitur

$$y = x^{\frac{2}{3}}(\alpha e^{ct} + \beta e^{-ct}) + \frac{x}{3c}(\alpha e^{ct} - \beta e^{-ct}).$$

Cafu autem $cc = -bb$ aequationis

$$ddy + bbx^{-\frac{8}{3}} y dx^2 = 0$$

integrale eft

$$y = x^{\frac{2}{3}}(\alpha \operatorname{cof.} bt + \beta \operatorname{fin.} bt) - \frac{x}{3b}(\beta \operatorname{cof.} bt - \alpha \operatorname{fin.} bt)$$

Exemplum 5.

949. *Integrale aequationis* $ddy - ccx^{-\frac{8}{5}} y dx^2 = 0$. *inuenire.*

Ob $\lambda = \frac{2}{5}$, eft $B = \frac{-3A}{5c}$ et $C = \frac{+3A}{5^2cc}$, hinc

$$z = Ax^{\frac{2}{5}} - \frac{3A}{5c}x^{\frac{1}{5}} + \frac{3A}{5^2cc} \text{ ideoque}$$

$$P = x^{\frac{2}{5}} + \frac{3}{5^2cc} \text{ et } Q = \frac{-3}{5cc}x^{\frac{1}{5}}.$$

Pofito ergo $t = 5x^{\frac{1}{5}}$ integrale erit

$$y = (x^{\frac{2}{5}} + \frac{3}{5^2cc})(\alpha e^{ct} + \beta e^{-ct}) - \frac{3}{5c}x^{\frac{1}{5}}(\alpha e^{ct} - \beta e^{-ct})$$

casu autem $cc = -bb$ aequationis

$$ddy + bbx^{-\frac{8}{3}}ydx^2 = 0$$

integrale est

$$y = \left(x^{\frac{2}{3}} - \frac{t}{3^2 bb}\right)(\alpha \operatorname{cos}.bt + \beta \operatorname{sin}.bt) + \frac{1}{3b}x^{\frac{1}{3}}(\beta \operatorname{cos}.bt - \alpha \operatorname{sin}.bt).$$

Problema 119.

950. Aequationis differentio-differentialis

$$ddy + ccx^{\frac{-4i}{2i-1}}ydx^2 = 0$$

integrale completum assignare, denotante i numerum integrum quemcunque.

Solutio.

Sit breuitatis gratia $t = -(2i-1)x^{\frac{-1}{2i-1}}$, vnde fit $x^{\frac{1}{2i-1}} = -\frac{(2i-1)}{t}$, ac posito $y = e^{ct}z$, pro valore ipsius z per seriem inuento

$$z = Ax^{\frac{i}{2i-1}} + Bx^{\frac{i+1}{2i-1}} + Cx^{\frac{i+2}{2i-1}} + Dx^{\frac{i+3}{2i-1}} \text{ etc.}$$

ob $\lambda = \frac{+i}{2i-1}$ hi coefficientes ita determinantur

$$B = \frac{i(i-1)A}{2(2i-1)c}; \; C = \frac{(i+1)(i-2)B}{4(2i-1)c}; \; D = \frac{(i+2)(i-3)C}{6(2i-1)c} \text{ etc.}$$

quibus substitutis, et introducto valore $x^{\frac{1}{2i-1}} = \frac{-(2i-1)}{t}$ erit

$$z = Ax^{\frac{i}{2i-1}}\left(1 - \frac{i(i-1)}{2ct} + \frac{i(ii-1)(i-2)}{2 \cdot 4cctt} - \frac{i(ii-1)(ii-4)(i-3)}{2 \cdot 4 \cdot 6 \cdot c^3t^3} + \text{etc.}\right)$$

siue

siue hoc modo

$$z = \frac{A}{t^i}\left(1 - \frac{i(i-1)}{2ct} + \frac{i(ii-1)(i-2)}{2\cdot 4cctt} - \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6c^3t^3} + \text{etc.}\right)$$

Atque hinc aequationis propositae integrale completum ita exprimetur:

$$y = t^{-i}\left(1 + \frac{i(ii-1)(i-2)}{2\cdot 4cctt} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8c^4t^4} + \text{etc.}\right)(\alpha e^{ct} + \beta e^{-ct})$$
$$- t^{-i}\left(\frac{i(i-1)}{2ct} + \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6c^3t^3} + \text{etc.}\right)(\alpha e^{ct} - \beta e^{-ct})$$

vbi in vtraque progressione lex formationis singulorum terminorum est manifesta.

Coroll. 1.

951. Hinc quoque illius aequationis

$$ddy + bbx^{\frac{-4i}{2i-1}}ydx^2 = 0$$

integrale completum manente $t = -(2i-1)x^{\frac{-1}{2i-1}}$ est

$$y = t^{-i}\left(1 - \frac{i(ii-1)(i-2)}{2\cdot 4bbtt} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8b^4t^4} - \text{etc.}\right)(\alpha\cos bt + \beta\sin bt)$$
$$+ t^{-i}\left(\frac{i(i-1)}{2bt} - \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6b^3t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2\cdot 4\cdot 6\cdot 8\cdot 10b^5t^5} - \text{etc.}\right)(\beta\cos bt - \alpha\sin bt).$$

Coroll. 2.

952. Si i sit numerus negatiuus haec integratio perinde succedit, aequationis enim

$$ddy - ccx^{\frac{-4i}{2i+1}}ydx^2 = 0$$

 posito

posito $t = (2i+1)x^{\frac{1}{2i+1}}$ integrale erit

$$y = t^i\left(1 + \frac{i(ii-1)(i+2)}{2\cdot 4\, cc\, tt} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8\, c^4 t^4} + \text{etc.}\right)(\alpha e^{ct} + \beta e^{-ct})$$

$$- t^i\left(\frac{i(i+1)}{2ct} + \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6\, c^3 t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10\, c^5 t^5} + \text{etc.}\right)(\alpha e^{ct} - \beta e^{-ct}).$$

Coroll. 3.

953. Simili modo huius aequationis

$$ddy + bbx^{\frac{-4i}{2i+1}}y\,dx^2 = 0$$

posito $t = (2i+1)x^{\frac{1}{2i+1}}$ integrale completum erit:

$$y = t^i\left(1 - \frac{ii(ii-1)(i+2)}{2\cdot 4\, bb\, tt} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8\, b^4 t^4} - \text{etc.}\right)(\alpha \operatorname{cof.} bt + \beta \operatorname{fin} bt)$$

$$+ t^i\left(\frac{i(i+1)}{2bt} - \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6\, b^3 t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10\, b^5 t^5} - \text{etc.}\right)(\beta \operatorname{cof.} bt - \alpha \operatorname{fin.} bt).$$

Coroll. 4.

954. In formulis finus et cofinus continentibus fi ponatur

$$\alpha = C \operatorname{fin.} \zeta \text{ et } \beta = C \operatorname{cof.} \zeta,$$

expreffiones noftrae ita contrahuntur, vt fiat

$$\alpha \operatorname{cof.} bt + \beta \operatorname{fin.} bt = C \operatorname{fin.} (bt + \zeta) \text{ et}$$

$$\beta \operatorname{cof.} bt - C\alpha \operatorname{fin.} bt = C \operatorname{cof.} (bt + \zeta),$$

vt iam hic C et ζ fint conftantes arbitrariae integrale completum reddentes.

Scholion.

Scholion.

955. Hinc egregium adipiscimur adminiculum ad casus integrabilitatis huius aequationis differentialis primi gradus

$$du + uudx + ax^n dx = 0$$

agnoscendos simulque integralia completa definienda; nascitur enim haec aequatio ex ista

$$ddy + ax^n y dx^2 = 0 \text{ ponendo } y = e^{\int u dx}$$

vnde ex illa vicissim haec oritur ponendo $u = \frac{dy}{ydx}$. Cum igitur istius integrale assignare licuerit casibus quibus exponens $n = \frac{-4i}{2i \pm 1}$ iisdem casibus aequationem differentialem primi gradus assignare licebit, vbi quidem duos casus euolui conuenit, prout a fuerit vel numerus negatiuus $a = -cc$ vel positiuus $a = +bb$. Hos igitur duos casus pertractasse operae erit pretium.

Problema 120.

956. Denotante i numerum integrum siue positiuum siue negatiuum quemcunque, inuenire integrale huius aequationis:

$$du + uudx - ccx^{\frac{-4i}{2i+1}} dx = 0.$$

Solutio.

Posito $u = \frac{dy}{ydx}$ haec aequatio transformatur in istam:

$$ddy - ccx^{\frac{-4i}{2i+1}} y dx^2 = 0$$

sumto

ſumto elemento dx conſtante, cuius integrale aſſignauimus. Poſito ſcilicet $t=(2i+1)x^{\frac{1}{2i+1}}$ eſt:

$$y=(\alpha e^{ct}+\beta e^{-ct})\left(t^i+\frac{i(ii-1)(i+2)}{2\cdot 4\, cc}t^{i-2}+\frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8\, c^4}t^{i-4}+\text{etc.}\right)$$
$$-(\alpha e^{ct}+\beta e^{-ct})\left(\frac{i(i+1)}{2c}t^{i-1}+\frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6\, c^3}t^{i-3}+\frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10\, c^5}t^{i-5}\ \text{etc.}\right)$$

Ponamus breuitatis gratia

$$y(\alpha e^{ct}+\beta e^{-ct})P-(\alpha e^{ct}-\beta e^{-ct})Q$$

et cum ſit

$$dt=x^{\frac{-2i}{2i+1}}dx \text{ ſeu } dx=x^{\frac{2i}{2i+1}}dt$$

erit

$$\frac{dy}{dx}=\frac{(\alpha e^{ct}+\beta e^{-ct})(dP-cQdt)+(\alpha e^{ct}-\beta e^{-ct})(cPdt-dQ)}{x^{\frac{2i}{2i+1}}dt}$$

ſiue

$$\frac{dy}{dx}=\frac{\alpha e^{ct}(dP+cPdt-dQ-cQdt)+\beta e^{-ct}(dP-cPdt+dQ-cQdt)}{x^{\frac{2i}{2i+1}}dt}.$$

At vero eſt

$$\frac{dP}{dt}=it^{i-1}+\frac{i(ii-1)(ii-4)}{2\cdot 4\, cc}t^{i-3}+\frac{i(ii-1)(ii-4)(ii-9)(ii-16)}{2\cdot 4\cdot 6\cdot 8\, c^4}t^{i-5}+\text{etc.}$$

$$cQ=\frac{i(i+1)}{2}t^{i-1}+\frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6\, c^2}t^{i-3}+\frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10\, c^4}t^{i-5}+\text{etc.}$$

$$cP=ct^i+\frac{i(ii-1)(i+2)}{2\cdot 4\, c}t^{i-2}+\frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8\, c^3}t^{i-4}+\text{etc.}$$

$$dQ=\frac{i(ii-1)}{2c}t^{i-2}+\frac{i(ii-1)(ii-4)(ii-9)}{2\cdot 4\cdot 6\, c^3}t^{i-4}+\text{etc.}$$

vnde

vnde colligitur:

$$\frac{dP-cQdt}{dt}=\frac{-i(i-1)}{2}t^{i-1}-\frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6c^2}t^{i-3}-\frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^4}t^{i-5}-\text{etc.}$$

$$\frac{cPdt-dQ}{dt}=ct^i+\frac{i(ii-1)(i-2)}{2\cdot 4}t^{i-2}+\frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8c^3}t^{i-4}+\text{etc.}$$

Ponamus ad abbreuiandum:

$$P=t^i+\frac{i(ii-1)(i+2)}{2\cdot 4cc}t^{i-2}+\frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8c^4}t^{i-4}+\text{etc.}$$

$$Q=\frac{i(i+1)}{2c}t^{i-1}+\frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6c^3}t^{i-3}+\frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^5}t^{i-5}+\text{etc.}$$

$$R=t^i+\frac{i(ii-1)(i-2)}{2\cdot 4cc}t^{i-2}+\frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8c^4}t^{i-4}+\text{etc.}$$

$$S=\frac{i(i-1)}{2c}t^{i-1}+\frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6c^3}t^{i-3}+\frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^5}t^{i-5}+\text{etc.}$$

vt fit

$$\frac{dy}{dx}=\frac{(\alpha e^{ct}+\beta e^{-ct})(-cS)+(\alpha e^{ct}-\beta e^{-ct})(cR)}{x^{\frac{2i}{2i+1}}}.$$

Quare cum fit $u=\frac{dy}{ydx}$ erit nostrae aequationis integrale completum:

$$\frac{a}{c}x^{\frac{2i}{2i+1}}u=\frac{(\alpha e^{ct}-\beta e^{-ct})R-(\alpha e^{ct}+\beta e^{-ct})S}{(\alpha e^{ct}+\beta e^{-ct})P-(\alpha e^{ct}-\beta e^{-ct})Q}\text{ siue}$$

$$\frac{a}{c}x^{\frac{2i}{2i+1}}u=\frac{\alpha e^{ct}(R-S)-\beta e^{-ct}(R+S)}{\alpha e^{ct}(P-Q)+\beta e^{-ct}(P+Q)}$$

quod ob rationem constantium $\alpha:\beta$ arbitrariam est completum.

Coroll. 1.

957. Quaternae formulae P, Q, R, S quae singulae casibus quibus i est numerus integer, abrum-

pan-

pantur, ita a se inuicem pendent, vt sit primo

$$R = P - \frac{dQ}{cdt} \text{ et } S = Q - \frac{dP}{cdt};$$

tum vero

$$dP + dR = \frac{\pm 2iRdt}{t} \text{ et } dQ + dS = \frac{2iSdt}{t}.$$

Coroll. 2.

958. Posito ergo vel $\alpha = 0$ vel $\beta = 0$, integralia particularia algebraica aequationis

$$du + uudx - ccx^{\frac{-4i}{2i+1}}dx = 0$$

exhiberi possunt, quae sunt

$$\tfrac{1}{c}x^{\frac{2i}{2i+1}}u = \frac{R-S}{P-Q} \text{ et } \tfrac{1}{c}x^{\frac{2i}{2i+1}}u = \frac{-S-R}{P+Q},$$

ideoque hac vna formula comprehendi possunt

$$\tfrac{1}{c}x^{\frac{2i}{2i+1}}u = \frac{-S \pm R}{P \pm Q}.$$

Scholion 1.

959. Pro variis ergo valoribus numeri i tam quantitas t quam litterae P, Q, R, S sequenti modo se habebunt: Primo scilicet si $i = 0$ erit $t = x$, atque $P = 1$, $Q = 0$, $R = 1$ et $S = 0$, reliquos casus in sequenti tabella repraesentemus:

$i = -1$; $t = -\frac{1}{x}$	$i = 1$; $t = 3x^{\frac{1}{3}}$
$P = \frac{1}{t}$; $Q = 0$	$P = t$; $Q = \frac{1}{c}$
$R = \frac{1}{t}$; $S = \frac{1}{ctt}$	$R = t$; $S = 0$

$i = -2$;

$i=-2;\ t=-\frac{3}{x^{\frac{1}{3}}}$	$i=2;\ t=5x^{\frac{1}{5}}$
$P=\frac{1}{t^2};\quad Q=\frac{1}{ct^3}$	$P=t^2+\frac{3}{cc};\ Q=\frac{3}{c}t$
$R=\frac{1}{t^3}+\frac{3}{cct^4};\ S=\frac{3}{ct^3}$	$R=t^3;\ S=\frac{1}{c}t$

$i=-3;\ t=\frac{-5}{x^{\frac{1}{5}}}$	$i=3;\ t=7x^{\frac{1}{7}}$
$P=\frac{1}{ct^3}+\frac{1.3}{cct^5}$	$P=t^3+\frac{1.3}{cc}t$
$Q=\frac{3}{ct^4}$	$Q=\frac{6}{c}tt+\frac{3.5}{c^3}$
$R=\frac{1}{t^3}+\frac{3.5}{cct^5}$	$R=t^3+\frac{1.3}{cc}t$
$S=\frac{6}{ct^4}+\frac{3.5}{c^3t^6}$	$S=\frac{3}{c}tt$

$i=-4;\ t=-\frac{7}{x^{\frac{1}{7}}}$	$i=4;\ t=9x^{\frac{1}{9}}$
$P=\frac{1}{t^4}+\frac{3.5}{cct^6}$	$P=t^4+\frac{3.3.5}{cc}t^2+\frac{3.5.7}{c^4}$
$Q=\frac{6}{ct^5}+\frac{3.5}{c^3t^7}$	$Q=\frac{10}{c}t^3+\frac{3.5.7}{c^3}t$
$R=\frac{1}{t^4}+\frac{3.3.5}{cct^6}+\frac{3.5.7}{c^4t^8}$	$R=t^4+\frac{3.5}{cc}t^2$
$S=\frac{10}{ct^5}+\frac{3.5.7}{c^7t^7}$	$S=\frac{6}{c}t^3+\frac{3.5}{c^3}t$

$i=-5;\; t=\frac{-9}{x^{\frac{1}{9}}}$

$P=\frac{1}{t^5}+\frac{3\cdot3\cdot5}{cct^7}+\frac{3\cdot5\cdot7}{c^4t^9}$

$Q=\frac{10}{ct^6}+\frac{3\cdot5\cdot7}{c^3t^8}$

$R=\frac{1}{t^5}+\frac{3\cdot5\cdot7}{cct^7}+\frac{3\cdot5\cdot7\cdot9}{c^4t^9}$

$S=\frac{15}{ct^6}+\frac{4\cdot3\cdot5\cdot7}{c^3t^8}+\frac{3\cdot5\cdot7\cdot9}{c^5t^{10}}$

$i=5;\; t=11x^{\frac{1}{11}}$

$P=t^5+\frac{3\cdot5\cdot7}{cc}t^3+\frac{3\cdot5\cdot7\cdot9}{c^4}t$

$Q=\frac{15}{c}t^4+\frac{4\cdot3\cdot5\cdot7}{c^3}t^2+\frac{5\cdot5\cdot7\cdot9}{c^5}$

$R=t^5+\frac{3\cdot3\cdot5}{cc}t^3+\frac{3\cdot5\cdot7}{c^4}t$

$S=\frac{10}{c}t^4+\frac{3\cdot5\cdot7}{c^3}t^2.$

$i=-6;\; t=\frac{-11}{x^{\frac{1}{11}}}$

$P=\frac{1}{t^6}+\frac{3\cdot5\cdot7}{cct^8}+\frac{3\cdot5\cdot7\cdot9}{c^4t^{10}}$

$Q=\frac{15}{ct^7}+\frac{4\cdot3\cdot5\cdot7}{c^3t^9}+\frac{3\cdot5\cdot7\cdot9}{c^5t^{11}}$

$R=\frac{1}{t^6}+\frac{2\cdot3\cdot5\cdot7}{cct^8}+\frac{5\cdot3\cdot5\cdot7\cdot9}{c^4t^{10}}+\frac{3\cdot5\cdot7\cdot9\cdot11}{c^6t^{12}}$

$S=\frac{21}{ct^7}+\frac{4\cdot5\cdot7\cdot9}{c^3t^9}+\frac{3\cdot5\cdot7\cdot9\cdot11}{c^5t^{11}}$

$i=6;\; t=13x^{\frac{1}{13}}$

$P=t^6+\frac{2\cdot3\cdot5\cdot7}{cc}t^4+\frac{5\cdot3\cdot5\cdot7\cdot9}{c^4}t^2+\frac{3\cdot5\cdot7\cdot9\cdot11}{c^6}$

$Q=\frac{21}{c}t^5+\frac{4\cdot5\cdot7\cdot9}{c^3}t^3+\frac{3\cdot5\cdot7\cdot9\cdot11}{c^5}t$

$R=t^6+\frac{3\cdot5\cdot7}{cc}t^4+\frac{3\cdot5\cdot7\cdot9}{c^4}t^2$

$S=\frac{15}{c}t^5+\frac{4\cdot3\cdot5\cdot7}{c^3}t^3+\frac{3\cdot5\cdot7\cdot9}{c^5}t.$

Scholion 2.

Scholion 2.

960. Dum hae formulae diligentius considerantur, noua se prodit relatio inter valores litterarum P, Q, R, S, quae in hoc consistit, vt perpetuo sit $PR - QS = t^{2i}$, cuius veritas primo quidem per inductionem deprehenditur, tum vero etiam per relationes supra datas demonstrari potest. Si enim valores

$$R = P - \frac{dQ}{cdt} \text{ et } S = Q - \frac{dP}{cdt}$$

in aequationibus:

$$dP + dR = \frac{2iRdt}{t} \text{ et } dQ + dS = \frac{2iSdt}{t}$$

substituantur, oriuntur hae duae aequationes:

$$2dP - \frac{ddQ}{cdt} = \frac{2iPdt}{t} - \frac{2idQ}{ct} \text{ et}$$

$$2dQ - \frac{ddP}{cdt} = \frac{2iQdt}{t} - \frac{2idP}{ct}$$

quarum illa per P, haec vero per $-Q$ multiplicata iunctim dant:

$$2PdP - 2QdQ + \frac{QddP - PddQ}{cdt} = \frac{2idt}{t}(PP - QQ) + \frac{2i}{ct}(QdP - PdQ).$$

Ponatur

$$PP - QQ = M \text{ et } \frac{QdP - PdQ}{cdt} = N \text{ erit}$$

$$dM + dN = \frac{2idt}{t}(M + N) \text{ seu } \frac{dM + dN}{M + N} = \frac{2idt}{t}$$

hincque integrando $M + N = Ct^{2i}$. At est

$$M + N = P\left(P - \frac{dQ}{cdt}\right) - Q\left(Q - \frac{dP}{cdt}\right) = PR - QS$$

 euidens

euidens autem eſt pro conſtante C vnitatem accipi debere.

Problema 121.

961. Denotante i numerum integrum ſiue poſitiuum ſiue negatiuum quemcunque, inuenire integrale completum huius aequationis

$$du + uudx + bbx^{\frac{-4i}{2i+1}}dx = 0.$$

Solutio.

Poſito $u = \frac{dy}{ydx}$ haec aequatio transformatur in iſtam

$$ddy + bbx^{\frac{-4i}{2i+1}}ydx^2 = 0$$

ſumto elemento dx conſtante, cuius integrale ſupra eſt aſſignatum. Scilicet poſito $t = (2i+1)x^{\frac{1}{2i+1}}$ invenimus (953, 954):

$$y = C\left(t^i - \frac{i(ii-1)(i+2)}{2\cdot 4bb}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8\cdot b^4}t^{i-4} - \text{etc.}\right)\text{ſin.}(bt+\zeta)$$
$$+ C\left(\frac{i(i+1)}{2b}t^{i-1} - \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6b^3}t^{i-3} + \text{etc.}\right)\text{coſ.}(bt+\zeta)$$

cuius loco breuitatis gratia ſcribamus:

$$y = CP\,\text{ſin.}(bt+\zeta) + CQ\,\text{coſ.}(bt+\zeta).$$

Hinc ob

$$dt = x^{\frac{-2i}{2i+1}}dx \text{ ſeu } dx = x^{\frac{2i}{2i+1}}dt$$

erit

erit

$$\frac{dy}{dx} = \frac{C(dP - bQdt)\sin.(bt+\zeta) + C(dQ + bPdt)\cos.(bt+\zeta)}{x^{\frac{2i}{2i+1}}\, dt}$$

Vnde cum sit $u = \frac{dy}{ydx}$ erit

$$u = \frac{(dP - bQdt)\sin.(bt+\zeta) + (dQ + bPdt)\cos.(bt+\zeta)}{x^{\frac{2i}{2i+1}}\, dt\,(P\sin.(bt+\zeta) + Q\cos.(bt+\zeta))}.$$

Ponamus

$$P + \frac{dQ}{bdt} = R \text{ et } Q - \frac{dP}{bdt} = S,$$

vt sit

$$\tfrac{1}{b} x^{\frac{2i}{2i+1}} u = \frac{R\cos.(bt+\zeta) - S\sin.(bt+\zeta)}{P\sin.(bt+\zeta) + Q\cos.(bt+\zeta)}$$

erit:

$$P = t^i - \frac{i(ii-1)(i+2)}{2.4\,bb} t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8\,b^4} t^{i-4} - \text{etc.}$$

$$Q = \frac{i(i+1)}{2b} t^{i-1} - \frac{i(ii-1)(ii-4)(i+3)}{2.4.6\,b^3} t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10\,b^5} t^{i-5} - \text{etc.}$$

$$R = t^i - \frac{i(ii-1)(i-2)}{2.4\,bb} t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8\,b^4} t^{i-4} - \text{etc.}$$

$$S = \frac{i(i-1)}{2b} t^{i-1} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6\,b^3} t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2.4.6.8.10\,b^5} t^{i-5} - \text{etc.}$$

atque ob angulum ζ introductum hoc integrale erit completum.

Coroll. 1.

962. Quaternarum ergo litterarum P, Q, R, S valores ita a se inuicem pendent vt sit primo:

$$R = P + \frac{dQ}{bdt} \text{ et } S = Q - \frac{dP}{bdt},$$

tum

tum vero etiam patet fore

$$dP+dR=\frac{2iRdt}{t} \text{ et } dQ+dS=\frac{2iSdt}{t}.$$

Coroll. 2.

963. Deinde etiam colligitur fore $PR+QS=t^{4i}$ quae aequalitas ex praecedentis problematis formulis deducitur, sumto $cc=-bb$ vbi Q et S abeunt in $Q\sqrt{-1}$ et $S\sqrt{-1}$.

Coroll. 3.

964. Hic casus a praecedente etiam hoc differt, quod hic nulla dentur integralia particularia algebraica. Quicunque enim valor angulo constanti ζ tribuatur, integrale semper sinum et cosinum cuiusdam anguli inuoluit.

Scholion 1.

965. Cum igitur aequationis

$$du+uudx+bbx^{\frac{-4i}{2i+1}}dx=0$$

integrale completum posito $t=(2i+1)x^{\frac{1}{2i+1}}$ fit

$$\frac{1}{b}x^{\frac{2i}{2i+1}}u=\frac{R\,\text{cos.}(bt+\zeta)-S\,\text{sin.}(bt+\zeta)}{P\,\text{sin.}(bt+\zeta)+Q\,\text{cos.}(bt+\zeta)}$$

pro singulis valoribus numeri i quantitas t cum litteris P, Q, R, S ita se habebit. Primo si $i=0$ erit

erit $P=1$, $Q=0$, $R=1$ et $S=0$ item $t=x$, ita vt integrale fit $\frac{1}{b}u=\frac{\cos.(bx+\zeta)}{\sin.(bx+\zeta)}$; reliquos casus sequens tabella exhibet:

$i=-1$; $t=-\frac{1}{x}$ $P=\frac{1}{t}$; $Q=0$ $R=\frac{1}{t}$; $S=\frac{1}{btt}$	$i=1$; $t=3x^{\frac{1}{3}}$ $P=t$; $Q=\frac{1}{b}$ $R=t$; $S=0$
$i=-2$; $t=\frac{-3}{x^{\frac{1}{3}}}$ $P=\frac{1}{tt}$; $Q=\frac{1}{bt^3}$ $R=\frac{1}{tt}-\frac{1}{bbt^4}$; $S=\frac{1}{bt^3}$	$i=2$; $t=5x^{\frac{1}{5}}$ $P=tt-\frac{1}{bb}$; $Q=\frac{1}{b}t$ $R=tt$; $S=\frac{1}{b}t$
$i=-3$; $t=\frac{-5}{x^{\frac{1}{5}}}$ $P=\frac{1}{t^3}-\frac{1\cdot 3}{bbt^5}$ $Q=\frac{3}{bt^4}$; $R=\frac{1}{t^3}-\frac{3\cdot 5}{bbt^5}$ $S=\frac{6}{bt^4}-\frac{3\cdot 5}{b^3t^6}$	$i=3$; $t=7x^{\frac{1}{7}}$ $P=t^3-\frac{3\cdot 5}{bb}t$ $Q=\frac{6}{b}tt-\frac{3\cdot 5}{b^3}$ $R=t^3-\frac{1\cdot 3}{bb}t$ $S=\frac{3}{b}tt$
$i=-4$; $t=\frac{-7}{x^{\frac{1}{7}}}$ $P=\frac{1}{t^4}-\frac{3\cdot 5}{bbt^6}$ $Q=\frac{6}{bt^5}-\frac{3\cdot 5}{b^3t^7}$ $R=\frac{1}{t^4}-\frac{3\cdot 3\cdot 5}{bbt^6}+\frac{3\cdot 5\cdot 7}{b^4t^8}$ $S=\frac{10}{bt^5}-\frac{3\cdot 5\cdot 7}{b^3t^7}$	$i=4$; $t=9x^{\frac{1}{9}}$ $P=t^4-\frac{3\cdot 3\cdot 5}{bb}tt+\frac{3\cdot 5\cdot 7}{b^4}$ $Q=\frac{10}{b}t^3-\frac{3\cdot 5\cdot 7}{b^3}t$ $R=t^4-\frac{3\cdot 5}{bb}tt$ $S=\frac{6}{b}t^3-\frac{3\cdot 5}{b^3}t$

$i=-5$;

$$i=-5;\; t=\frac{-9}{x^{\frac{1}{9}}}$$

$$P=\frac{1}{t^5}-\frac{3\cdot 5\cdot 5}{bbt^7}+\frac{3\cdot 5\cdot 7}{b^4t^9}$$

$$Q=\frac{10}{bt^6}-\frac{5\cdot 5\cdot 7}{b^3t^8}$$

$$R=\frac{1}{t^5}-\frac{3\cdot 5\cdot 7}{bbt^7}+\frac{3\cdot 5\cdot 7\cdot 9}{b^4t^9}$$

$$S=\frac{15}{bt^6}-\frac{4\cdot 3\cdot 5\cdot 7}{b^3t^8}+\frac{3\cdot 5\cdot 7\cdot 9}{b^5t^{10}}$$

$$i=5;\; t=11x^{\frac{1}{11}}$$

$$P=t^5-\frac{3\cdot 5\cdot 7}{bb}t^3+\frac{3\cdot 5\cdot 7\cdot 9}{b^4}t$$

$$Q=\frac{15}{b}t^4-\frac{4\cdot 3\cdot 5\cdot 7}{b^3}t^2+\frac{3\cdot 5\cdot 7\cdot 9}{b^5}$$

$$R=t^5-\frac{3\cdot 5\cdot 5}{bb}t^3+\frac{3\cdot 5\cdot 7}{b^4}t$$

$$S=\frac{10}{b}t^4-\frac{3\cdot 5\cdot 7}{b^2}t^2$$

$$i=-6;\; t=\frac{-11}{x^{\frac{1}{11}}}$$

$$P=\frac{1}{t^6}-\frac{3\cdot 5\cdot 7}{bbt^8}+\frac{3\cdot 5\cdot 7\cdot 9}{b^4t^{10}}$$

$$Q=\frac{15}{bt^7}-\frac{4\cdot 3\cdot 5\cdot 7}{b^3t^9}+\frac{3\cdot 5\cdot 7\cdot 9}{b^5t^{11}}$$

$$R=\frac{1}{t^6}-\frac{2\cdot 3\cdot 5\cdot 7}{bbt^8}+\frac{5\cdot 3\cdot 5\cdot 7\cdot 9}{b^4t^{10}}-\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{b^6t^{12}}$$

$$S=\frac{21}{bt^7}-\frac{4\cdot 5\cdot 7\cdot 9}{b^3t^9}+\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{b^5t^{11}}$$

$$i=6;\; t=13x^{\frac{1}{13}}$$

$$P=t^6-\frac{2\cdot 3\cdot 5\cdot 7}{bb}t^4+\frac{5\cdot 3\cdot 5\cdot 7\cdot 9}{b^4}tt-\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{b^6}$$

$$Q=\frac{21}{b}t^5-\frac{4\cdot 5\cdot 7\cdot 9}{b^3}t^3+\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{b^5}t$$

$$R=t^6-\frac{3\cdot 5\cdot 7}{bb}t^4+\frac{3\cdot 5\cdot 7\cdot 9}{b^4}tt$$

$$S=\frac{15}{b}t^5-\frac{4\cdot 3\cdot 5\cdot 7}{b^3}t^3+\frac{3\cdot 5\cdot 7\cdot 9}{b^5}t.$$

Scho-

Scholion 2.

966. Forma integralis inuenta modum suppeditat aequationem propositam

$$du + uudx + Ax^{\frac{-4i}{2i+1}}dx = 0$$

in speciem simpliciorem transformandi. Primo enim ponatur

$$x^{\frac{2i}{2i+1}}u = v \text{ seu } u = x^{\frac{-2i}{2i+1}}v,$$

ac prodibit

$$x^{\frac{-2i}{2i+1}}dv - \frac{2i}{2i+1}x^{\frac{-4i-1}{2i+1}}vdx + x^{\frac{-4i}{2i+1}}vvdx + Ax^{\frac{-4i}{2i+1}}dx = 0$$

seu

$$dv - \frac{2i}{2i+1}\cdot\frac{vdx}{x} + x^{\frac{-2i}{2i+1}}vvdx + Ax^{\frac{-2i}{2i+1}}dx = 0.$$

Ponatur porro $t = (2i+1)x^{\frac{1}{2i+1}}$, erit

$$dt = x^{\frac{-2i}{2i+1}}dx \text{ et } \frac{dt}{t} = \frac{1}{2i+1}\cdot\frac{dx}{x},$$

vnde fit

$$dv - \frac{2ivdt}{t} + vvdt + Adt = 0.$$

Sit insuper $v = \frac{i}{t} + z$ vt prodeat

$$-\frac{idt}{tt} + dz - \frac{2iidt}{tt} - \frac{2izdt}{t} + \frac{iidt}{tt} + \frac{2izdt}{t} + zzdt + Adt = 0$$

seu $dz + zzdt - \frac{i(i+1)dt}{tt} + Adt = 0$

quae ergo quoties i eſt numerus integer, eſt integrabilis. Simili modo haec aequatio

$$du + uudx + Ax^n dx = 0$$

generalius ita transformari poteſt: poſito $u = x^\lambda v$ et $v = z - \frac{1}{2}\lambda x^{-\lambda-1}$ obtinetur:

$$dz + x^\lambda zzdx + \tfrac{1}{4}\lambda(\lambda+2)x^{-\lambda-2}dx + Ax^{n-\lambda}dx = 0$$

quae porro poſito $x^\lambda dx = dt$ ſeu $x^{\lambda+1} = (\lambda+1)t$ abit in

$$dz + zzdt + \frac{\lambda(\lambda+2)dt}{4(\lambda+1)^2 tt} + A(\lambda+1)^{\frac{n-2\lambda}{\lambda+1}} t^{\frac{n-2\lambda}{\lambda+1}} dt = 0$$

quae aequatio eſt integrabilis, quoties $n = \frac{-4i}{2i+1}$, vnde numerum λ pro lubitu aſſumendo innumerabiles formae exhiberi poſſunt. Si capiatur $\lambda = -1$ fit $t = lx$ et $dz + zzdt - \frac{1}{4}dt + Ae^{(n+2)t}dt = 0$.

CAPVT VIII.

CAPVT VIII.

DE ALIARVM AEQVATIONVM DIFFERENTIO-DIFFERENTIALIVM RESOLVTIONE PER SERIES INFINITAS.

Problema 122.

967.

Formam generalem aequationum differentio-differentialium, quas commode per ſeries reſoluere licet, exhibere, earumque integralia inueſtigare.

Solutio.

Primo alias aequationes commode per ſeries reſoluere non licet, niſi in quibus altera variabilis y cum ſuis differentialibus dy et ddy nusquam plus vna dimenſione obtinet; quoniam pro y ſeriem infinitam ſubſtituendo in calculos nimis moleſtos incideremus, ſi vsquam plures dimenſiones ingrederentur. Huiusmodi ergo aequationes in hac forma

$$ddy + Mdxdy + Nydx^2 = Xdx^2$$

continentur. Tum vero vt ſeriei pro y aſſumtae quili-

quilibet terminus per solum praecedentem determinetur, qui est casus resolutionis maxime notabilis, duplicis tantum generis terminos ratione alterius variabilis x inesse oportet, siquidem ad dimensiones, quas ipsa x cum suo differentiali dx constituit, respiciamus. Vnde primo quidem reiecto termino Xdx^2 aequationes hoc modo resolubiles in hac forma continentur:

$$xx(a+bx^n)ddy+x(c+ex^n)dxdy+(f+gx^n)ydx^2=0.$$

Pro cuius resolutione fingamus:

$$y=Ax^\lambda+Bx^{\lambda+n}+Cx^{\lambda+2n}+Dx^{\lambda+3n}+\text{etc.}$$

et facta substitutione sequens serierum summa ad nihilum redigi debet:

$$\begin{array}{llll}
\lambda(\lambda-1)Aax^\lambda & +(\lambda+n)(\lambda+n-1)Bax^{\lambda+n} & +(\lambda+2n)(\lambda+2n-1)Cax^{\lambda+2n} & +\text{etc.} \\
 & +\lambda(\lambda-1)Ab & +(\lambda+n)(\lambda+n-1)Bb & \\
+\lambda Ac & +(\lambda+n)Bc & +(\lambda+2n)Cc & \\
 & +\lambda Ae & +(\lambda+n)Be & \\
+Af & +Bf & +Cf & \\
 & +Ag & +Bg. &
\end{array}$$

Hic ergo primo exponens λ ita accipi debet, vt sit

$$\lambda(\lambda-1)a+\lambda c+f=0$$

tum vero pro reliquis fieri oportet

$$((\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f)B=-(\lambda(\lambda-1)b+\lambda e+g)A$$
$$((\lambda+2n)(\lambda+2n-1)a+(\lambda+2n)c+f)C=-((\lambda+n)(\lambda+n-1)b+(\lambda+n)e+g)B$$
$$((\lambda+3n)(\lambda+3n-1)a+(\lambda+3n)c+f)D=-((\lambda+2n)(\lambda+2n-1)b+(\lambda+2n)e+g)C$$
etc.

Cum

Cum igitur ſit

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

ſi ponamus breuitatis cauſa

$$\lambda(\lambda-1)b+\lambda e+g=h$$

erit

$$(n'n+2\lambda-1)a+nc)B=-hA$$

$$(2n(2n+2\lambda-1)a+2nc)C=-(n'n+2\lambda-1)b+ne+h)B$$

$$(3n(3n+2\lambda-1)a+3nc)D=-(2n(2n+2\lambda-1)b+2ne+h)C$$

etc.

Quia ergo niſi $a=0$, pro λ gemini inueniuntur valores ſcilicet,

$$\lambda=\frac{a-c\pm\sqrt{((a-c)^2-4af)}}{2a},$$

binae ſeries pro y inueniuntur, quae vtcunque combinatae integrale completum aequationis propoſitae praebent.

Aliter.

Propoſita aequatione hac:

$$xx(a+bx^n)ddy+x(c+ex^n)dxdy+(f+gx^n)ydx^2=0$$

ſeries quoque ordine retrogrado fingi poteſt

$$y=Ax^{\lambda}+Bx^{\lambda-n}+Cx^{\lambda-2n}+Dx^{\lambda-3n}+\text{etc.}$$

vnde

vnde oritur ad nihilum reducendum

$$
\begin{array}{llll}
+\lambda(\lambda-1)Abx^{\lambda+n} & +(\lambda-n)(\lambda-n-1)Bbx^{\lambda} & +(\lambda-2n)(\lambda-2n-1)Cbx^{\lambda-n} & \text{etc.} \\
 & +\lambda(\lambda-1)Aa & +(\lambda-n)(\lambda-n-1)Ba & \\
+\lambda Ae & +(\lambda-n)Be & +(\lambda-2n)Ce & \\
 & +\lambda Ac & +(\lambda-n)Bc & \\
+Ag & +Bg & +Cg & \\
 & +Af & +Bf. &
\end{array}
$$

Hic ergo exponentem λ ita accipi oportet, vt fiat

$$\lambda(\lambda-1)b+\lambda e+g=0.$$

Tum vero si ponamus

$$\lambda(\lambda-1)a+\lambda c+f=h$$

determinatio coefficientium ita se habebit:

$$n((n-2\lambda+1)b-e)B=-hA$$

$$2n((2n-2\lambda+1)b-e)C=-(n(n-2\lambda+1)a+nc+h)B$$

$$3n((3n-2\lambda+1)b-e)D=-(2n(2n-2\lambda+1)a-2nc+h)C$$

etc.

Coroll. 1.

968. Ex priore solutione, si i denotet numerum integrum positiuum, series assumta alicubi abrumpetur, si fuerit

$$in(in+2\lambda-1)b+ine+h=0 \text{ vel}$$

$$(\lambda+in)(\lambda+in-1)b+(\lambda+in)e+g=0$$

hoc

hoc eft

$$\left.\begin{array}{l}\lambda\lambda b+\lambda(2in-1)b+in(in-1)b\\ +\lambda e \qquad\qquad +ine+g\end{array}\right\}=0.$$

Coroll. 2.

969. Aequatio ergo noftra integrationem admittit, fi litterae f et g ita fuerint comparatae, vt fit

$$f=-\lambda(\lambda-1)a-\lambda c \text{ et}$$

$$g=-(\lambda+in)(\lambda+in-1)b-(\lambda+in)e.$$

Vel fumtis duobus numeris μ et ν, vt fit $\nu-\mu$ diuifibile per exponentem n fi fuerit

$$f=-\mu(\mu-1)a-\mu c \text{ et } g=-\nu(\nu-1)b-\nu e.$$

Coroll. 3.

970. Cum hinc fit

$$\mu=\frac{a-c\pm\sqrt{((a-c)^2-4af)}}{2a} \text{ et } \nu=\frac{b-e\pm\sqrt{((b-e)^2-4bg)}}{2b}$$

aequatio habebit integrale algebraicum, fi fuerit $\nu-\mu=in$ denotante i numerum integrum pofitiuum: hoc eft fi fit

$$in=\frac{c}{2a}-\frac{e}{2b}\pm\frac{\sqrt{((b-e)^2-4bg)}}{2b}\mp\frac{\sqrt{((a-c)^2-4af)}}{2a}.$$

Coroll. 4.

971. Pro ferie autem inuenienda fi eueniat, vt exponens λ fiat imaginarius, notari conuenit effe

$$x^{\alpha+\beta\sqrt{-1}}=x^{\alpha}\cdot e^{\beta\sqrt{-1}\cdot lx}=x^{\alpha}(\text{cof.}\,\beta lx+\sqrt{-1}\cdot\text{fin.}\,\beta lx)$$

 vnde

vnde binae feries ita combinari poterunt, vt integrale confequatur formam realem.

Scholion.

972. Vtraque folutio generatim fpectata duplicem feriem pro variabili y fuppeditat, pro gemino exponentis λ valore, quarum combinatio integrale completum exhibet. Solutio fcilicet prior pro exponente λ hos duos praebet valores

$$\lambda = \frac{a-c \pm \surd((a-c)^2 - 4af)}{2a},$$

folutio vero pofterior

$$\lambda = \frac{b-e \pm \surd((b-e)^2 - 4bg)}{2b},$$

ita vt hoc modo integrale completum duplici modo exprimi poffit; quae binae formae etiamfi maxime diuerfae, atque adeo interdum altera per exponentes imaginarios progrediatur, dum altera habet reales, tamen fibi aequipollentes effe debent. Quin etiam euenire poteft, vt altera folutio vel etiam vtraque ad integrale completum exhibendum fit inepta, dum vnicam feriem fuppeditet. Incommodum hoc pro vtraque folutione duplici modo accidere poteft; pro priori nempe folutione, vbi exponentem λ ex hac aequatione:

$$\lambda(\lambda-1)a + \lambda c + f = 0$$

definiri oportet, vnicus inde pro λ eruitur valor, fi fuerit vel $a = 0$, vel $4af = (a-c)^2$, priori cafu tantum fit $\lambda = -\frac{f}{c}$; altero ipfius λ valore quafi in infi-

infinitum abeunte. Posteriori casu vero ambo ipsius λ valores fiunt inter se aequales scilicet $\lambda = \frac{a-c}{2a}$. Idem incommodum in altera solutione locum habet, si fuerit vel $b=0$ vel $4bg=(b-e)^2$, vnde patet fieri posse, vt altera solutio huiusmodi incommodo laboret, dum altera eo careat, quin etiam vt vtraque eodem inquinetur. Quocirca ostendi conueniet, quemadmodum etiam his casibus integrale completum inuestigari debeat; quorsum etiam casum referamus, quo ambo ipsius λ valores fiunt imaginarii quandoquidem ad imaginariam speciem tollendam singulari artificio est opus. Denique vero etiam binae series pro y exhibendae difficultate premuntur, quoties bini valores ipsius λ differentiam habent per exponentem n diuisibilem, quorum casuum euolutio etiam explicari meretur.

Problema 123.

973. Proposita aequatione differentio-differentiali:

$$xx(a+bx^n)ddy+x(c+ex^n)dx\,dy+(f+gx^n)y\,dx^2=0$$

si eueniat, vt binae series ascendentes pro y assumtae vel in vnam coalescant, vel altera fiat impossibilis, integrale completum per series exprimere.

Solutio.

Assumta serie

$$y=Ax^\lambda+Bx^{\lambda+n}+Cx^{\lambda+2n}+Dx^{\lambda+3n}+\text{etc.}$$

 si

si eueniat vt bini valores ipsius λ ex aequatione

$$\lambda(\lambda-1)a+\lambda c+f=0$$

vel fiant aequales, vel differentiam per n diuisibilem obtineant, valor ipsius y praeter potestates ipsius x etiam logarithmum ipsius x inuoluet. Quare pro aequationis resolutione statim ponamus $y=u+vlkx$, vt sit $y=u+vlx+\alpha v$, denotante α quantitatem constantem quamcunque. Hinc erit

$$dy=du+\frac{vdx}{x}+dvlkx \text{ atque}$$

$$ddy=ddu+\frac{2dxdv}{x}-\frac{vdx^2}{xx}+ddvlkx$$

quibus valoribus substitutis, aequatio nostra hanc induet formam

$$\left.\begin{array}{r} xx(a+bx^n)ddu+2x(a+bx^n)dxdv-(a+bx^n)vdx^2 \\ +x(c+ex^n)dxdu+(c+ex^n)vdx^2 \\ +(f+gx^n)udx^2 \\ +(xx(a+bx^n)ddv+x(c+ex^n)dxdv+(f+gx^n)vdx^2)lkx \end{array}\right\}=0$$

vbi partem postremam logarithmo affectam seorsim nihilo aequari oportet. Quare posito

$$v=Ax^\lambda+Bx^{\lambda+n}+Cx^{\lambda+2n}+Dx^{\lambda+3n}+\text{etc.}$$

exponenti λ ex aequatione

$$\lambda(\lambda-1)a+\lambda c+f=0$$

is valor tribuatur, qui nulli incommodo est obnoxius; eritque pro reliquis coefficientibus ponendo

$$\lambda(\lambda-1)b+\lambda e+g=nb$$

vt fequitur:

$$\begin{aligned}
&((n+2\lambda-1)a+c)B+bA=0\\
&2((2n+2\lambda-1)a+c)C+(n+2\lambda-1)b+e)B+bB=0\\
&3((3n+2\lambda-1)a+c)D+2((2n+2\lambda-1)b+e)C+bC=0\\
&4((4n+2\lambda-1)a+c)E+3((3n+2\lambda-1)b+e)D+bD=0\\
&\text{etc.}
\end{aligned}$$

His coefficientibus ita definitis, quorum primus A arbitrio noftro relinquitur, ponamus

$$u=\Delta+\mathfrak{A}x^{\lambda}+\mathfrak{B}x^{\lambda+n}+\mathfrak{C}x^{\lambda+2n}+\mathfrak{D}x^{\lambda+3n}+\text{etc.}$$

qui valor fi in priori aequatione cum ferie pro v inuenta fubftituatur, fequentes feries ad nihilum reduci oportet

$$\begin{array}{llll}
(a+bx^n)\frac{dd\Delta}{dx^2} & +\lambda(\lambda-1)\mathfrak{A}ax^{\lambda} & +(\lambda+n)(\lambda+n-1)\mathfrak{B}ax^{\lambda+n} & +(\lambda+2n)(\lambda+2n-1)\mathfrak{C}ax^{\lambda+2n}\ \text{etc.}\\
(c+ex^n)\frac{d\Delta}{dx} & & +\lambda(\lambda-1)\mathfrak{A}b & +(\lambda+n)(\lambda+n-1)\mathfrak{B}b\\
 & +\lambda\mathfrak{A}c & +(\lambda+n)\mathfrak{B}c & +(\lambda+2n)\mathfrak{C}c\\
(f+gx^n)\Delta & & +\lambda\mathfrak{A}e & +(\lambda+n)\mathfrak{B}e\\
 & +\mathfrak{A}f & +\mathfrak{B}f & +\mathfrak{C}f\\
 & & +\mathfrak{A}g & +\mathfrak{B}g\\
 & +2\lambda Aa & +2(\lambda+n)Ba & +2(\lambda+2n)Ca\\
 & +A(c-a) & +2\lambda Ab & +2(\lambda+n)Bb\\
 & & +B(c-a) & +C(c-a)\\
 & & +A(e-b) & +B(e-b)
\end{array}$$

 Cum

Cum autem sit

$$\lambda(\lambda-1)a+\lambda c+f=0 \text{ et } \lambda(\lambda-1)b+\lambda e+g=nb$$

expressio haec transmutabitur in hanc formam

$$xx(a+bx^n)\frac{dd\Delta}{dx^2}+x(c+ex^n)\frac{d\Delta}{dx}+(f+gx^n)\Delta$$

$$\begin{array}{lll} ((2\lambda-1)a+c)\mathrm{A}x^\lambda & +((2n+2\lambda-1)a+c)\mathrm{B}x^{\lambda+n} & +((4n+2\lambda-1)a+c)\mathrm{C}x^{\lambda+2n} \\ & +((2\lambda-1)b+e)\mathrm{A} & +((2n+2\lambda-1)b+e)\mathrm{B} \\ & +n((n+2\lambda-1)a+c)\mathfrak{B} & +2n((2n+2\lambda-1)a+e)\mathfrak{C} \\ & +nb\mathfrak{A} & +n((n+2\lambda-1)b+e)\mathfrak{B} \\ & & +nb\mathfrak{B} \end{array}$$

vbi Δ denotat quosdam terminos seriei

$$\mathfrak{A}x^\lambda+\mathfrak{B}x^{\lambda+n} \text{ etc.}$$

praemittendos, ita vt ordine retrogrado sit

$$\Delta=\mathfrak{a}x^{\lambda-n}+\mathfrak{b}x^{\lambda-2n}+\mathfrak{c}x^{\lambda-3n}+\ldots+\mathfrak{j}x^{\lambda-in}.$$

Quod principium quomodo quouis casu sit constituendum, sequentia sunt obseruanda.

I. Principium hoc locum habere nequit, nisi fuerit

$$(\lambda-in)(\lambda-in-1)a+(\lambda-in)c+f=0,$$

cum igitur sit

$$\lambda(\lambda-1)a+\lambda c+f=0$$

inde erit

$$\lambda=in+\frac{a-c-\sqrt{((a-c)^2-4af)}}{2a},$$

hinc vero

$$\lambda=\frac{a-c+\sqrt{((a-c)^2-4af)}}{2a},$$

quan-

quandoquidem hi duo valores conuenire nequeunt, nisi ibi signum radicale negatiue, hic vero positiue accipiatur. Aequatis autem his valoribus fit

$$in = \tfrac{1}{a}\sqrt{((a-c)^2 - 4af)}, \text{ seu}$$

$$iinnaa = (a-c)^2 - 4bf \text{ hincque}$$

$$f = \frac{(a-c)^2}{4a} - \tfrac{1}{4}iinna,$$

vnde fit

$$\lambda = \frac{a-c}{2a} + \tfrac{1}{2}in.$$

Quare si aequatio proposita ita fuerit comparata, vt sit

$$ina = \sqrt{((a-c)^2 - 4af)},$$

tum sumto

$$\lambda = \frac{a-c}{2a} + \tfrac{1}{2}in,$$

ac pro v sumta serie

$$v = \mathrm{A}x^{\lambda} + \mathrm{B}x^{\lambda+n} \text{ etc.}$$

alteram seriem u ita constitui conuenit

$$u = \mathfrak{j}x^{\lambda-in} + \ldots\ldots + \mathfrak{a}x^{\lambda-n} + \mathfrak{A}x^{\lambda} + \mathfrak{B}x^{\lambda+n} + \mathfrak{C}x^{\lambda+2n} \text{ etc.}$$

Hic est casus, quo bini valores ipsius λ ex aequatione

$$\lambda(\lambda-1)a + \lambda c + f = 0$$

differentiam habent per n diuisibilem, vbi notandum seriem v a maiore valore ipsius λ, seriem vero u a minore inchoari debere.

II.

II. Principium Δ omitti nequit nisi fuerit

$$(2\lambda - 1)a + c = 0,$$

quo casu fit $\lambda = \frac{a-c}{2a}$: atque hic est casus, quo aequationis

$$\lambda(\lambda - 1)a + \lambda c + f = 0$$

binae radices fiunt inter se aequales, ideoque $f = \frac{(a-c)^2}{4a}$. Continetur ergo hic casus in praecedente sumendo ibi $i = 0$. Quare hoc modo resoluentur casus, quibus bini valores ipsius λ vel sunt inter se aequales, vel differentiam habent per exponentem n diuisibilem. Sicque reperitur integrale completum per duas series ascendentes v et u expressum, quarum illa v per lx multiplicatur.

Coroll. 1.

974. Quando ergo in aequatione proposita coefficientes a, c et f ita sunt comparati, vt aequationis $\lambda(\lambda-1)a + \lambda c + f = 0$ radices sint $\lambda = \mu$ et $\lambda = \mu - in$, denotante i numerum integrum positiuum, integrale completum huiusmodi habebit formam $y = u + av + vlx$.

Coroll. 2.

975. Hic autem binae quantitates v et u ex his aequationibus

$$\text{I. } xx(a+bx^n)ddv + x(c+ex^n)dxdv + (f+gx^n)vdx^2 = 0$$

$$\text{II. } \left.\begin{aligned} xx(a+bx^n)ddu + x(c+ex^n)dxdu + (f+gx^n)udx^2 \\ +2x(a+bx^n)dxdv - (a+bx^n)vdx^2 \\ +(c+ex^n)vdx^2 \end{aligned}\right\} = 0$$

ita

ita determinari poterunt, vt ponatur:

$$v = A x^{\mu} + B x^{\mu+n} + C x^{\mu+2n} + D x^{\mu+3n} + \text{etc.}$$
$$u = \mathfrak{A} x^{\mu-in} + \mathfrak{B} x^{\mu-in+n} + \mathfrak{C} x^{\mu-in+2n} + \mathfrak{D} x^{\mu-in+3n} \text{ etc.}$$

Has ſcilicet ſeries ſubſtituendo omnes coefficientes ex vno definire licebit.

Scholion.

976. Logarithmo ergo ipſius x in ſubſidium vocato his caſibus, quos commemorauimus, integrale completum aequationis propoſitae per ſeries aſcendentes exhiberi poteſt, dum ſine hoc artificio integrale tantum particulare inuenitur. Quando enim aequatio $\lambda(\lambda-1)a+\lambda c+f=0$ duas radices habet, quarum differentia per exponentem n eſt diuiſibilis puta $\lambda=\mu$ et $\lambda=\mu-in$, priore methodo ſola ſeries, quae incipit a poteſtate x^{μ}, determinari poteſt; ſi enim altera a poteſtate $x^{\mu-in}$ incipiens pro y aſſumeretur, coefficiens cuiusdam termini reperiretur infinitus, vnde ſequentes omnes forent quoque infiniti, quod incommodum introducendo logarithmus ipſius x feliciter tollitur. Hunc igitur vſum iſtius reſolutionis aliquot exemplis illuſtraſſe iuuabit.

Exemplum 1.

977. *Aequationis differentio-differentialis:*

$$xddy + dxdy + gx^{n-1}ydx^2 = 0$$

integrale completum per ſeries aſcendentes exhibere.

Hanc aequationem ad nostram formam reducendo habebimus:

$$xxddy + xdxdy + gx^n ydx^2 = 0,$$

vbi ergo est $a=1$; $b=0$; $c=1$; $e=0$ et $f=0$. Hinc $\lambda(\lambda-1)+\lambda=0$ seu $\lambda\lambda=0$, ita vt bini valores ipsius λ sint aequales et $=0$. Quare posito $y=u+av+vlx$, resolui oportet has aequationes:

I. $xxddv + xdxdv + gx^n vdx^2 = 0$ et

II. $\begin{aligned} xxddu + xdxdu + gx^n udx^2 &= 0 \\ +2xdxdv. & \end{aligned}$

Statuamus ergo

$$v = A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc. et}$$

$$u = \mathfrak{A} + \mathfrak{B}x^n + \mathfrak{C}x^{2n} + \mathfrak{D}x^{3n} + \text{etc.}$$

ac prior aequatio praebet

$$\left.\begin{array}{llll} n(n-1)Bx^n & +2n(2n-1)Cx^{2n} & +3n(3n-1)Dx^{3n} & +\text{etc.} \\ +nB & +2nC & +3nD & \\ +Ag & +Bg & +Cg & \end{array}\right\} = 0$$

vnde fit

$$B = \frac{-Ag}{nn};\ C = \frac{-Bg}{4nn};\ D = \frac{-Cg}{9nn};\ E = \frac{-Dg}{16nn} \text{ etc.}$$

Tum vero altera aequatio dat

$$\left.\begin{array}{llll} nn\mathfrak{B}x^n & +4nn\mathfrak{C}x^{2n} & +9nn\mathfrak{D}x^{3n} & +\text{etc.} \\ +\mathfrak{A}g & +\mathfrak{B}g & +\mathfrak{C}g & \\ +2nB & +4nC & +6nD & \end{array}\right\} = 0$$

vnde

vnde colligitur:

$$\mathfrak{B} = \frac{-\mathfrak{A}g}{nn} - \frac{2B}{n}; \quad \mathfrak{C} = \frac{-\mathfrak{B}g}{4nn} - \frac{2C}{2n}; \quad \mathfrak{D} = \frac{-\mathfrak{C}g}{9nn} - \frac{2D}{3n} \text{ etc.}$$

Hic autem tuto assumere licet $\mathfrak{A} = 0$, quoniam termini ex $\mathfrak{A}$ oriundi, continentur in membro αv. Cum igitur sit

$$B = \frac{-Ag}{n^2}; \quad C = \frac{+Ag^2}{1.4n^4}; \quad D = \frac{-Ag^3}{1.4.9n^6}; \quad E = \frac{+Ag^4}{1.4.9.16n^8} \text{ etc.}$$

erit vt sequitur

$$\mathfrak{B} = \frac{2Ag}{n^3}; \quad \mathfrak{C} = \frac{-2Agg}{4n^5} - \frac{2Agg}{2.1.4n^5} = \frac{-6Agg}{2.1.4n^5}$$

$$\mathfrak{D} = \frac{6Ag^3}{2.1.4.9n^7} + \frac{2Ag^3}{3.1.4.9n^7} = \frac{22Ag^3}{2.3.1.4.9n^7}$$

$$\mathfrak{E} = \frac{-22Ag^4}{2.3.1.4.9.16n^9} - \frac{2Ag^4}{4.1.4.9.16n^9} = \frac{-100Ag^4}{2.3.4.1.4.9.16n^9}$$

$$\mathfrak{F} = \frac{100Ag^5}{2.3.4.1.4.9.16.25n^{11}} + \frac{2Ag^5}{5.1.4.9.16.25n^{11}} = \frac{548Ag^5}{2.3.4.5.1.4.9.16.25n^{11}}$$

sicque obtinentur sequentes valores:

$$\mathfrak{B} = \frac{2Ag}{n^3}; \quad \mathfrak{C} = \frac{-6Ag^2}{1.8n^5}; \quad \mathfrak{D} = \frac{22Ag^3}{1.8.27n^7}; \quad \mathfrak{E} = \frac{-100Ag^4}{1.8.27.64n^9}$$

$$\mathfrak{F} = \frac{548Ag^5}{1.8.27.64.125n^{11}}; \quad \mathfrak{G} = \frac{-3528Ag^6}{1.8.27.64.125.216n^{13}} \text{ etc.}$$

vbi numeratores 2, 6, 22, 100, 548, 3528 etc. singuli ita per binos praecedentes definiuntur

$6 = 3.2 - 1.0$; $22 = 5.6 - 4.2$; $100 = 7.22 - 9.6$;

$548 = 9.100 - 16.22$; $3528 = 11.548 - 25.100$ etc.

Consequenter integrale ita exprimetur

$$y = \frac{2Ag}{n^3}x^n - \frac{6Ag^2}{1.8n^5}x^{2n} + \frac{22Ag^3}{1.8.27n^7}x^{3n} - \frac{100Ag^4}{1.8.27.64n^9}x^{4n} + \text{etc.}$$

$$+ A\left(1 - \frac{g}{nn}x^n + \frac{g^2}{1.4n^4}x^{2n} - \frac{g^3}{1.4.9n^6}x^{2n} + \frac{g^4}{1.4.9.16n^8}x^{4n} - \text{etc.}\right)lx$$

$$+ \alpha - \frac{\alpha g}{nn}x^n + \frac{\alpha g^2}{1.4n^4}x^{2n} - \frac{\alpha g^2}{1.4.9n^6}x^{3n} + \frac{\alpha g^4}{1.4.9.16n^8}x^{4n} - \text{etc.}$$

vbi A et α sunt binae constantes arbitrariae.

Exemplum 2.

978. *Aequationis differentio - differentialis*

$$x(1-xx)ddy-(1+xx)dxdy+xydx^2=0$$

integrale completum per series ascendentes assignare.

Ad formam nostram reducta est

$$xx(1-xx)ddy-x(1+xx)dxdy+xxydx^2=0$$

ita vt sit $n=2$; $a=1$; $b=-1$; $c=-1$; $e=-1$, $f=0$ et $g=1$, vnde aequationis $\lambda(\lambda-1)-\lambda=0$ radices sint $\lambda=0$ et $\lambda=2$, quarum differentia per $n=2$ diuisa dat 1. Posito ergo $y=u+\alpha v+vlx$, statui debet

$$v=Ax^2+Bx^4+Cx^6+Dx^8+\text{etc. et}$$
$$u=\mathfrak{A}+\mathfrak{B}x^2+\mathfrak{C}x^4+\mathfrak{D}x^6+\mathfrak{E}x^8\text{ etc.}$$

quae series ex sequentibus aequationibus determinari debent:

I. $xx(1-xx)ddv-x(1+xx)dxdv+xxvdx^2=0$

II. $\left.\begin{array}{r} xx(1-xx)ddu-x(1+xx)dxdu+xxudx^2 \\ +2x(1-xx)dxdv-2vdx^2 \end{array}\right\}=0.$

Hinc pro prioris determinatione fit:

$$\left.\begin{array}{rrrr} 2Ax^2 & +12Bx^4 & +30Cx^6 & +56Dx^8 \\ & -2A & -12B & -30C \\ -2A & -4B & -6C & -8D\text{ etc.} \\ & -2A & -4B & -6C \\ & +A & +B & +C \end{array}\right\}=0$$

ideoque

ideoque

$$2.4B=1.3A;\quad 4.6C=3.5B;\quad 6.8D=5.7C \text{ etc.}$$

feu

$$B=\tfrac{1.3}{2.4}A;\quad C=\tfrac{1.3.3.3}{2.4.4.6}A;\quad D=\tfrac{1.3.3.5.5.7}{2.4.4.6.6.8}A \text{ etc.}$$

Ex altera vero aequatione reperitur:

$$\left.\begin{array}{rrrr}
2\mathfrak{B}x^2 & +12\mathfrak{C}x^4 & +30\mathfrak{D}x^6 & +56\mathfrak{E}x^8+\text{etc.}\\
 & -2\mathfrak{B} & -12\mathfrak{C} & -30\mathfrak{D}\\
-2\mathfrak{B} & -4\mathfrak{C} & -6\mathfrak{D} & -8\mathfrak{E}\\
 & -2\mathfrak{B} & -4\mathfrak{C} & -6\mathfrak{D}\\
+\mathfrak{A} & +\mathfrak{B} & +\mathfrak{C} & +\mathfrak{D}\\
+4A & +8B & +12C & +16D\\
 & -4A & -8B & -12C\\
-2A & -2B & -2C & -2D
\end{array}\right\}=0$$

vnde fieri oportet:

$$\mathfrak{A}+2A=0;\quad 2.4\mathfrak{C}-1.3\mathfrak{B}+6B-4A=0;\quad 4.6\mathfrak{D}-3.5\mathfrak{C}$$
$$+10C-8B=0;\ 6.8\mathfrak{E}-5.7\mathfrak{D}+14D-12C=0 \text{ etc.}$$

feu cum fit

$$B=\tfrac{1.3}{2.4}A;\quad C=\tfrac{3.5}{4.6}B;\quad D=\tfrac{5.7}{4.6}C \text{ etc.}$$

erit $\mathfrak{A}=-2A$ tum vero

$$2.\ 4\mathfrak{C}-1.3\mathfrak{B}-\tfrac{2.7}{2.4}A=0;\quad \mathfrak{C}=\tfrac{1.3}{2.4}\mathfrak{B}+\tfrac{2.7}{2^2.4^2}A$$

$$4.\ 6\mathfrak{D}-3.5\mathfrak{C}-\tfrac{2.21}{4.6}B=0;\quad \mathfrak{D}=\tfrac{3.5}{4.6}\mathfrak{C}+\tfrac{2.21}{4^2.6^2}B$$

$$6.\ 8\mathfrak{E}-5.7\mathfrak{D}-\tfrac{2.45}{6.8}C=0;\quad \mathfrak{E}=\tfrac{5.7}{6.8}\mathfrak{D}+\tfrac{2.45}{6^2.8^2}C$$

$$8.\ 10\mathfrak{F}-7.9\mathfrak{E}-\tfrac{2.77}{8.10}D=0;\quad \mathfrak{F}=\tfrac{7.9}{8.10}\mathfrak{E}+\tfrac{2.77}{8^2.10^2}D.$$

Dum ergo capiatur $\mathfrak{A} = -2A$, littera $\mathfrak{B}$ pro lubitu accipi poteſt; nihilque impedit, quo minus nihilo aequalis ſtatuatur, ſiquidem conſtans α ſupra eſt inducta.

Exemplum 3.

979. *Aequationis differentio-differentialis:*

$$xx(1+bxx)ddy+x(-5+exx)dxdy+(5+gxx)ydx^2=0$$

integrale completum per ſeries aſcendentes exhibere.

Quia hic eſt $a=1$; $c=-5$ et $f=5$, aequatio $\lambda(\lambda-1)-5\lambda+5=0$ ſeu $\lambda\lambda-6\lambda+5=0$ radices habet $\lambda=1$ et $\lambda=5$, quarum differentia 4 per $n=2$ diuidi poteſt. Poſito ergo $y=u+\alpha v+v\,lx$ ſtatuatur

$$v=Ax^5+Bx^7+Cx^9+Dx^{11}+Ex^{13} \text{ etc. et}$$
$$n=\mathfrak{A}x+\mathfrak{B}x^3+\mathfrak{C}x^5+\mathfrak{D}x^7+\text{ etc.}$$

aequationes reſoluendae erunt:

$$\text{I. } xx(1+bxx)ddv+x(-5+exx)dxdv+(5+gxx)vdx^2=0 \text{ et}$$

$$\text{II. } \left.\begin{array}{r} xx(1+bxx)ddu+x(-5+exx)dxdu+(5+gxx)udx^2 \\ +2x(1+bxx)dxdv-(1+bxx)vdx^2 \\ +(-5+exx)vdx^2 \end{array}\right\}=0$$

vbi prior ducit ad:

$$\left.\begin{array}{llll} 5.4Ax^5 & +7.6Bx^7 & +9.8Cx^9 & +11.10Dx^{11} \text{ etc.} \\ -5.5A & -5.7B & -5.9C & -5.11D \\ +5A & +5B & +5C & +5D \\ & +5.4Ab & +7.6Bb & +9.8Cb \\ & +5Ae & +7Be & +9Ce \\ & +Ag & +Bg & +Cg \end{array}\right\}=0$$

poſte-

posterior vero ad:

$$\left.\begin{array}{lllll}
 & +2.3\mathfrak{B}x^3 & +4.5\mathfrak{C}x^5 & +6.7\mathfrak{D}x^7 & +8.9\mathfrak{E}x^9 \text{ etc.}\\
-5\mathfrak{A}x & -5.3\mathfrak{B} & -5.5\mathfrak{C} & -5.7\mathfrak{D} & -5.9\mathfrak{E}\\
+5\mathfrak{A} & +5\mathfrak{B} & +6\mathfrak{C} & +5\mathfrak{D} & +5\mathfrak{E}\\
 & & +2.3\mathfrak{B}b & +4.5\mathfrak{C}b & +6.7\mathfrak{D}b\\
 & +\mathfrak{A}e & +3\mathfrak{B}e & +5\mathfrak{C}e & +7\mathfrak{D}e\\
 & +\mathfrak{A}g & +\mathfrak{B}g & +\mathfrak{C}g & +\mathfrak{D}g\\
 & & +2.5A & +2.7B & +2.9C\\
 & & -6A & -6B & -6C\\
 & & & +2.5Ab & +2.7Bb\\
 & & & -Ab & -Bb\\
 & & & +Ae & +Be
\end{array}\right\} = 0$$

Inde fit	seu
$12B+A(20b+5e+g)=0$	$2.6B+A(4.5b+5e+g)=0$
$32C+B(42b+7e+g)=0$	$4.8C+B(6.7b+7e+g)=0$
$60D+C(72b+9e+g)=0$	$6.10D+C(8.9b+9e+g)=0$
etc.	etc.

Hinc autem

$$-4\mathfrak{B}+\mathfrak{A}(e+g)=0$$

$$0\mathfrak{C}+\mathfrak{B}(2.3b+2e+g)+4A=0$$

$$2.6\mathfrak{D}+\mathfrak{C}(4.5b+5e+g)+8B+A(9b+e)=0$$

$$4.8\mathfrak{E}+\mathfrak{D}(6.7b+7e+g)+12C+B(13b+e)=0$$

etc.

Ex

Ex prioribus formulis litterae B, C, D etc. per A determinantur, ex posteriorum vero secunda fit $\mathfrak{B} = \frac{-4\mathrm{A}}{2.\ b+3e+g}$ ex prima autem $\mathfrak{A} = \frac{4\mathfrak{B}}{e+g}$, tum vero $\mathfrak{C}$ pro lubitu assumi potest, indeque reliqui coefficientes $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ etc. definiuntur.

Scholion.

980. Exemplum hoc occasionem nobis suppeditat phaenomena quaedam singularia obseruandi. Scilicet etiamsi integrale completum ingenere lx inuoluat, tamen id a logarithmo liberum prodit certis casibus. Primo nempe si sit $g=-e$ fit $\mathfrak{B}=0$ manente $\mathfrak{A}$ indefinito, tum vero $bb\mathfrak{B}=0$ capi oportet $\mathrm{A}=0$, $\mathrm{C}=0$ etc. ideoque $v=0$. Porro vero erit

$$2.\quad 6\mathfrak{D}+4\mathfrak{C}(5b+e)=0$$
$$4.\quad 8\mathfrak{E}+6\mathfrak{D}(7b+e)=0$$
$$6.\quad 10\mathfrak{F}+8\mathfrak{E}(9b+e)=0$$

etc.

vbi $\mathfrak{C}$ altera est constans arbitraria, eritque aequationis

$$xx(1+bxx)ddy+x(-5+exx)dxdy+(5-exx)ydx^2=0$$

integrale completum:

$$y=\mathfrak{A}x+\mathfrak{C}x^5+\mathfrak{D}x^7+\mathfrak{E}x^9+\mathfrak{F}x^{11}\text{etc.}$$

quod adeo finite exprimitur si $e=-(2i+5)b$, pro i sumendo numeros 0, 1, 2, 3, 4 etc.

Secundo

Secundo si sit

$$2\cdot 3b + 3e + g = 0 \text{ seu } g = -6b - 3e$$

fit $\mathfrak{B} = -\frac{1}{2}\mathfrak{A}(3b+e)$, tum vero $A = 0$; $B = 0$, $C = 0$ etc. ergo $v = 0$. Porro vero reperitur

$$\mathfrak{D} = -\tfrac{1}{6}\mathfrak{C}(7b+e); \quad \mathfrak{E} = -\tfrac{1}{8}\mathfrak{D}(9b+e); \quad \mathfrak{F} = -\tfrac{1}{10}\mathfrak{E}(11b+e) \text{ etc.}$$

hincque

$$y = \mathfrak{A}x - \tfrac{1}{2}\mathfrak{A}(3b+e)x^3 + \mathfrak{C}x^5 + \mathfrak{D}x^7 + \mathfrak{E}x^9 + \text{etc.}$$

vbi $\mathfrak{A}$ et $\mathfrak{C}$ arbitrio nostro relinquuntur.

Tertio si sit

$$4\cdot 5b + 5e + g = 0 \text{ seu } g = -20b - 5e,$$

primo fit $B = 0$, $C = 0$, $D = 0$ etc. ideoque $v = Ax^5$, tum vero

$$\mathfrak{B} = -\mathfrak{A}(5b+e); \quad -\mathfrak{B}(14b+2e) + 4A = 0 \text{ seu } \mathfrak{B} = \frac{2A}{7b+e}$$

hincque $A = \frac{1}{2}\mathfrak{A}(5b+e)(7b+e)$ porro

$$2\cdot 6\mathfrak{D} + A(9b+e) = 0$$
$$4\cdot 8\mathfrak{E} + 2\mathfrak{D}(11b+e) = 0$$
$$6\cdot 10\mathfrak{F} + 2\mathfrak{E}(13b+e) = 0.$$
etc.

Per $\mathfrak{A}$ ergo definiuntur coefficientes $\mathfrak{B}$, A, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ etc. ac $\mathfrak{C}$ quoque arbitrio nostro relinquitur, vnde integrale completum hoc casu erit

$$y = Ax^5 lx + \mathfrak{C}x^5 + \mathfrak{A}x + \mathfrak{B}x^3 + * + \mathfrak{D}x^7 + \mathfrak{E}x^9 + \text{etc.}$$

quae expressio fit finita quoties $(2i+5)b + e = 0$.

Exemplum 4.

981. *Si in priori exemplo fit* $e = -7b$ *et* $g = 15b$, *aequationis*

$$xx(1+bxx)ddy - x(5+7bxx)dxdy + 5(1+3bxx)ydx^2 = 0$$

integrale completum algebraice exhibere.

Erit ergo $\mathfrak{B} = +2\mathfrak{A}b$, $A = 0$, $\mathfrak{D} = 0$, $\mathfrak{E} = 0$ ideoque $v = 0$ et $u = \mathfrak{A}x + 2\mathfrak{A}bx^3 + \mathfrak{C}x^5$, vnde pro $\mathfrak{A}$ et $\mathfrak{C}$ fumendo conftantes quascunque, erit integrale completum:

$$y = \mathfrak{A}x(1+2bxx) + \mathfrak{C}x^5.$$

Integralia ergo particularia erunt:

$$y = ax(1+2bxx);\ y = ax^5;\ y = ax(1+bxx)^2.$$

Coroll. 1.

982. Pofito $y = e^{\int z dx}$ vt fit $z = \frac{dy}{y dx}$ aequationis huius differentialis primi gradus:

$$xx(1+bxx)dz + xx(1+bxx)zzdx - x(5+7bxx)zdx + 5(1+3bxx)dx = 0$$

integrale completum eft $z = \frac{\mathfrak{A}(1+6bxx)+5\mathfrak{C}x^4}{\mathfrak{A}x(1+2bxx)+\mathfrak{C}x^5}$.

Coroll. 2.

983. Aequatio autem differentio-differentialis integrabilis redditur, fi diuidatur per $xx(1+bxx)^2$, eritque integrale

$$\frac{xdy - 5ydx}{x(1+bxx)} = Cdx \text{ feu } dy - \frac{5ydx}{x} = Cdx(1+bxx)$$

quae

quae per x^8 diuisa integrale praebet

$$\frac{y}{x^8} = \frac{-C}{4x^4} - \frac{C}{2x^2} + D \text{ seu}$$

$$y = -\tfrac{1}{4}Cx(1 + 2bxx) + Dx^8,$$

vt ante.

Scholion.

984. Deficit autem adhuc integratio completa nostrae aequationis generalis per series ascendentes, casu quo $a = 0$, ideoque $\lambda c + f = 0$, vnde vnicus pro exponente λ valor definitur $\lambda = \frac{-f}{c}$ qui tantum integrale particulare suppeditat, atque hoc etiam tollitur, si fuerit $c = 0$. Quia autem his casibus a est $= 0$, coefficiens b certo adsit necesse est, ex quo integrale completum per series descendentes exhiberi poterit, cum aequatio $\lambda(\lambda - 1)b + \lambda e + g = 0$ duas semper contineat radices, ex quibus duplex series obtinetur. Simile autem hic incommodum vsu venire potest, quando binae radices ipsius λ vel prodeunt aequales, vel differentiam habent per exponentem n diuisibilem. Verum huic incommodo seriem per lx multiplicatam introducendo simili methodo medela affertur, qua in hoc problemate sumus vsi, ac superfluum foret istam euolutionem hic repetere. Quodsi autem binae radices ipsius λ tam pro seriebus ascendentibus quam descendentibus fiant imaginariae, ostendendum restat, quomodo integrale completum per series infinitas exprimi oporteat.

Problema 124.

985. Propofita aequatione differentio - differentiali

$$xx(a+bx^n)ddy+x(c+ex^n)dxdy+(f+gx^n)ydx^2=0$$

fi eueniat vt aequatio

$$\lambda(\lambda-1)a+\lambda c+f=0$$

radices habeat imaginarias, eius integrale completum per feries afcendentes exhibere.

Solutio.

Ex fupra allatis (971) colligitur hoc cafu ftatui debere

$$y=v\,\text{fin.}\,\mu lx+u\,\text{cof.}\,\mu lx,$$

vnde fit

$$dy=(dv-\tfrac{\mu udx}{x})\,\text{fin.}\,\mu lx+(\tfrac{\mu vdx}{x}+du)\,\text{cof.}\,\mu lx \text{ et}$$

$$ddy=(ddv-\tfrac{2\mu dxdu}{x}+\tfrac{\mu udx^2}{xx}-\tfrac{\mu\mu vdx^2}{xx})\,\text{fin.}\,\mu lx$$

$$+(ddu+\tfrac{2\mu dxdv}{x}-\tfrac{\mu vdx^2}{xx}-\tfrac{\mu\mu udx^2}{xx})\,\text{cof.}\,\mu lx$$

qua facta fubftitutione, fi terminos tam fin. μlx quam cof μlx affectos feorfim ad nihilum redigamus, obtinebimus duas fequentes aequationes:

$$\text{I.}\quad \left.\begin{aligned} xx(a+bx^n)ddv+x(c+ex^n)dxdv+(f+gx^n)vdx^2 \\ -2\mu x(a+bx^n)dxdu-\mu\mu(a+bx^n)vdx^2 \\ +\mu(a+bx^n)udx^2 \\ -\mu(c+ex^n)udx^2 \end{aligned}\right\}=0$$

II.

$$\text{II.}\quad \left.\begin{array}{r} xx(a+bx^n)ddu+x(c+ex^n)dxdu+(f+gx^n)udx^2 \\ +2\mu x(a+bx^n)dxdv-\mu\mu(a+bx^n)udx^2 \\ -\mu(a+bx^n)vdx^2 \\ +\mu(c+ex^n)vdx^2 \end{array}\right\}=0$$

Iam pro v et u assumamus has series ascendentes

$$v=Ax^\lambda+Bx^{\lambda+n}+Cx^{\lambda+2n}+Dx^{\lambda+3n}+\text{etc.}$$

$$u=\mathfrak{A}x^\lambda+\mathfrak{B}x^{\lambda+n}+\mathfrak{C}x^{\lambda+2n}+\mathfrak{D}x^{\lambda+3n}+\text{etc.}$$

iisque substitutis prior aequatio abit in hanc:

$$\begin{array}{lll}
\lambda(\lambda-1)Aax^\lambda & +(\lambda+n)(\lambda+n-1)Bax^{\lambda+n} & +(\lambda+2n)(\lambda+2n-1)Cax^{\lambda+2n} \\
 & +\lambda(\lambda-1)Ab & +(\lambda+n)(\lambda+n-1)Bb \\
+\lambda Ac & +(\lambda+n)Bc & +(\lambda+2n)Cc \\
 & +\lambda Ae & +(\lambda+n)Be \\
+Af & +Bf & +Cf \\
 & +Ag & +Bg \\
-2\mu\lambda\mathfrak{A}a & -2\mu(\lambda+n)\mathfrak{B}a & -2\mu(\lambda+2n)\mathfrak{C}a \\
 & -2\mu\lambda\mathfrak{A}b & -2\mu(\lambda+n)\mathfrak{B}b \\
-\mu\mu Aa & -\mu\mu Ba & -\mu\mu Ca \\
 & -\mu\mu Ab & -\mu\mu Bb \\
+\mu\mathfrak{A}a & +\mu\mathfrak{B}a & +\mu\mathfrak{C}a \\
 & +\mu\mathfrak{A}b & +\mu\mathfrak{B}b \\
-\mu\mathfrak{A}c & -\mu\mathfrak{B}c & -\mu\mathfrak{C}c \\
 & -\mu\mathfrak{A}e & -\mu\mathfrak{B}e.
\end{array}$$

 Hinc

Hinc altera aequatio facile formatur permutandis litteris latinis et germanicis atque insuper signum numeri μ mutando.

Vtrinque ergo potestas prima x^λ exigit has aequationes:

$$A(\lambda(\lambda-1)a+\lambda c+f-\mu\mu a)-\mu\mathfrak{A}(2\lambda a-a+c)=0$$
$$\mathfrak{A}(\lambda(\lambda-1)a+\lambda c+f-\mu\mu a)+\mu A(2\lambda a-a+c)=0$$

vnde necesse est vt sit

tam $2\lambda a-a+c=0$

quam $\lambda(\lambda-1)a+\lambda c+f-\mu\mu a=0.$

Inde fit $\lambda=\frac{1}{2}-\frac{c}{2a}$, qui valor hic substitutus dat $-a(\frac{1}{4}-\frac{cc}{4aa})+\frac{c}{2}-\frac{cc}{2a}+f=\mu\mu a=-\frac{a}{4}+\frac{c}{2}-\frac{cc}{4a}+f$ seu $\mu\mu a=\frac{4af-(a-c)^2}{4a}$ ideoque $\mu=\frac{\sqrt{(4af-(a-c)^2)}}{2a}$ et $\lambda=\frac{a-c}{2a}$. Vnde patet hanc solutionem locum habere si $4af>(a-c)^2$, quo ipso casu praecedens solutio fiebat imaginaria. Hic autem quantitates A et $\mathfrak{A}$ arbitrio nostro relinquuntur.

Terminus vero $x^{\lambda+n}$ vtrinque postulat has aequationes:

$$B((\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a)+A(\lambda(\lambda-1)b+\lambda e+g-\mu\mu b)$$
$$-\mu\mathfrak{B}(2(\lambda+n)a-a+c)-\mu\mathfrak{A}(2\lambda b-b+e)=0 \text{ et}$$
$$\mathfrak{B}((\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a)+\mathfrak{A}(\lambda(\lambda-1)b+\lambda e+g-\mu\mu b)$$
$$+\mu B(2(\lambda+n)a-a+c)+\mu A(2\lambda b-b+e)=0.$$

Sit

Sit breuitatis gratia

$$(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a=\alpha$$
$$\lambda(\lambda-1)b+\lambda e+g-\mu\mu b=\beta;$$
$$2(\lambda+n)a-a+c=2na=\gamma;$$
$$2\lambda b-b+e=\delta$$

vt habeamus:

$$B\alpha+A\beta-\mu\mathfrak{B}\gamma-\mu\mathfrak{A}\delta=0 \text{ et}$$

$$\mathfrak{B}\alpha+\mathfrak{A}\beta+\mu B\gamma+\mu A\delta=0$$

vnde colligitur:

$$B=\frac{-A(\alpha\beta+\mu\mu\gamma\delta)+\mu\mathfrak{A}(\alpha\delta-\beta\gamma)}{\alpha\alpha+\mu\mu\gamma\gamma} \text{ et}$$

$$\mathfrak{B}=\frac{-\mathfrak{A}(\alpha\beta+\mu\mu\gamma\delta)-\mu A(\alpha\delta-\beta\gamma)}{\alpha\alpha+\mu\mu\gamma\gamma}.$$

At vero ex valoribus assumtis est

$\alpha=nna$; $\beta=\frac{(ae-bc)(a-c)}{2aa}-\frac{bf}{a}+g$; $\gamma=2na$; $\delta=\frac{ae-be}{a}$

vnde ex assumtis A et $\mathfrak{A}$ definiuntur B et $\mathfrak{B}$ hincque porro C, $\mathfrak{C}$, D, $\mathfrak{D}$ etc.

Exemplum 1.

986. *Sit* c=a *et* f=a, *vt fiat* $\mu=1$, *et inuestigetur integrale huius aequationis*

$$xx(a+bx^n)ddy+x(a+ex^n)dxdy+(a+gx^n)ydx^2=0.$$

Hic

Hic ergo erit $\lambda = 0$ et $\mu = 1$, vnde posito $y = v \sin. lx + u \cos. lx$; ac pro v et u sumtis seriebus

$$v = A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.}$$

$$u = \mathfrak{A} + \mathfrak{B}x^n + \mathfrak{C}x^{2n} + \mathfrak{D}x^{3n} + \text{etc.}$$

coefficientes A et $\mathfrak{A}$ pro lubitu accipi possunt. Ex iis primo ob $\alpha = nna$; $\beta = g - b$; $\gamma = 2na$ et $\delta = e - b$, erit

$$B = \frac{-A(nna(g-b)+2na(e-b)) + \mathfrak{A}(nna(e-b)-2na(g-b))}{n^4aa+4nnaa}$$

seu $B = \dfrac{-A(n(g-b)+2(e-b)) + \mathfrak{A}(n(e-b)-2(g-b))}{na(nn+4)}$

et $\mathfrak{B} = \dfrac{-\mathfrak{A}(n(g-b)+2(e-b)) - A(n(e-b)-2(g-b))}{na(nn+4)}$.

Pro sequentibus coefficientibus habebimus:

$$C(2n(2n-1)a + 2na + a) - a) + B(n(n-1)b + ne + g - b)$$
$$- \mathfrak{C}(4na - a + a) - \mathfrak{B}(2nb - b + e) = 0$$

seu $4nnCa + B((nn-n-1)b+ne+g) - 4n\mathfrak{C}a - \mathfrak{B}((2n-1)b+e) = 0$

et $4nn\mathfrak{C}a + \mathfrak{B}((nn-n-1)b+ne+g) + 4nCa + B((2n-1)b+e) = 0$

quarum illa per n multiplicata huic addatur, vt prodeat

$$4n(nn+1)Ca + B((n^3-nn+n-1)b+(nn+1)e+ng)$$
$$+ \mathfrak{B}(-(nn+1)b+g) = 0$$

hinc $C = \dfrac{-B((n-1)(nn+1)b+(nn+1)e+ng)+\mathfrak{B}((nn+1)b-g)}{4na(nn+1)}$

et $\mathfrak{C} = \dfrac{-\mathfrak{B}((n-1)(nn+1)b+(nn+1)e+ng)-B((nn+1)b-g)}{4na(nn+1)}$.

Porro

Porro erit

$$9nn\mathrm{D}a+\mathrm{C}((4nn-2n-1)b+2ne+g)-6n\mathfrak{D}a-\mathfrak{C}((4n-1)b+e)=0$$

$$9nn\mathfrak{D}a+\mathfrak{C}((4nn-2n-1)b+2ne+g)+6n\mathrm{D}a+\mathrm{C}((4n-1)b+e)=0$$

quarum illa per $3n$, haec vero per 2 multiplicata iunctim dant:

$$3n(9nn+4)\mathrm{D}a+\mathrm{C}((12n^3-6nn+5n-2)b+2(3nn+1)e+3ng)$$
$$+\mathfrak{C}((-4nn-n-2)b+ne+2g)=0$$

vnde sequitur

$$\mathrm{D}=\frac{-\mathrm{C}((12n^3-6nn+5n-2)b+2(3nn+1)e+3ng)+\mathfrak{C}((4nn+n+2)b-ne-2g)}{3n(9nn+4)a}$$

$$\mathfrak{D}=\frac{-\mathfrak{C}((12n^3-6nn+5n-2)b+2(3nn+1)e+3ng)-\mathrm{C}((4nn+n+2)b-ne-2g)}{3n(9nn+4)a}.$$

In genere autem ex coefficientibus quibuscunque M et $\mathfrak{M}$ sequentes N et $\mathfrak{N}$ definiuntur per has formulas:

$$in(iinn+4)\mathrm{N}a+\mathrm{M}((i(i-1)^2n^3-i(i-1)nn+(3i-4)n-2)b+i(i-1)nne+2e+ing$$
$$-\mathfrak{M}((2(i-1)nn+(i-2)n+2)b-(i-2)ne-2g)=0$$

$$in(iinn+4)\mathfrak{N}a+\mathfrak{M}((i(i-1)^2n^3-i(i-1)nn+(3i-4)n-2)b+i(i-1)nne+2e+ing$$
$$+\mathrm{M}((2(i-1)nn+(i-2)n+2)b-(i-2)ne-2g)=0.$$

Coroll. 1.

987. Si quantitates b, e, g ita sint comparatae, vt binae litterae sibi respondentes N et $\mathfrak{N}$ euanescant, sequentes omnes euanescent, et integrale

completum forma finita exprimetur. Ita vt B et $\mathfrak{B}$ euanescant, fieri debet

$$2(g-b)=n(e-b) \text{ et } n(g-b)=-2(e-b);$$

vnde fit $g=e=b$ et ipsa aequatio proposita factorem habebit $a+bx^n$.

Coroll. 2.

988. In genere autem integrale finite exprimetur, si denotante i numerum integrum quemcunque positiuum sit

$$g=((i-1)nn+\tfrac{1}{2}(i-2)n+1)b-\tfrac{1}{2}(i-2)ne,$$

tum vero

$$e=-(2(i-1)n-1),$$

vnde fit

$$g=((i-1)^2nn+1)b.$$

Exemplum 2.

989. *Sumto* $n=1$ *si sit* $e=-b$ *et* $g=2b$, *huius aequationis:*

$$xx(a+bx)ddy+x(a-bx)dxdy+(a+2bx)ydx^2=0$$

integrale completum assignare.

Ex formulis modo inuentis colligimus:

$$B=\frac{-A(g+2e-3b)+\mathfrak{A}(e+b-2g)}{5a}=\frac{3Ab-4\mathfrak{A}b}{5a} \text{ et}$$

$$\mathfrak{B}=\frac{3\mathfrak{A}b+4Ab}{5a}; \text{ tum vero}$$

$$\mathfrak{C}=\frac{-B(2e+g)+\mathfrak{B}(2b-g)}{2a}=0 \text{ et } \mathfrak{C}=0.$$

Quo-

Quocirca habebimus:

$$v = A + \frac{(3A - 4\mathfrak{A})b}{5a}x \quad \text{et} \quad u = \mathfrak{A} + \frac{(3\mathfrak{A} + 4A)b}{5a}x$$

hincque integrale completum elicitur:

$$y = A\sin.lx + \mathfrak{A}\cos.lx + \frac{bx}{5a}((3A - 4\mathfrak{A})\sin.lx + (3\mathfrak{A} + 4A)\cos.lx).$$

Coroll. 1.

990. Sumto $\mathfrak{A} = 0$ habebitur integrale particulare

$$y = A\left(\sin.lx + \frac{3bx}{5a}\sin.lx + \frac{4bx}{5a}\cos.lx\right)$$

sin autem sit $A = 0$ aliud habebitur:

$$y = \mathfrak{A}\left(\cos.lx - \frac{4bx}{5a}\sin.lx + \frac{3bx}{5a}\cos.lx\right).$$

Coroll. 2.

991. Posito $y = e^{\int s\,dx}$ aequatio nostra reducitur ad hanc

$$xx(a+bx)ds + xx(a+bx)ssdx + x(a-bx)sdx + (a+2bx)dx = 0$$

cuius integrale habetur $s = \frac{dy}{ydx}$ inde definiendum, quae aequatio in plures alias formas transfundi potest.

Scholion.

992. Simili modo integratio per series descendentes instituitur, si exponentes singulorum terminorum prodeant imaginarii; quod seorsim exposuisse ne

 opus

opus quidem erit. Atque haec sufficiunt, vt pateat, quibusnam cautelis in resolutione aequationum per series infinitas sit vtendum. Summus autem vsus istarum euolutionum in hoc consistit, vt aequationes differentio-differentiales exhiberi queant, quarum saltem integrale particulare algebraicum assignare liceat, quos casus supra §. 969. indicauimus. Similis porro integratio per series infinitas pari modo extendi potest ad huiusmodi aequationes:

$$xx(a+bx^n+\beta x^{2n})ddy + x(c+ex^n+\varepsilon x^{2n})dxdy + (f+gx^n+\gamma x^{2n})ydx^2 = 0$$

tum autem seriei quaesitae quilibet terminus per duos praecedentes determinatur, ita vt si bini contigui euanescant sequentes omnes in nihilum sint abituri. Quodsi autem terminus ab y vacuus affuerit, resolutio in series fit facilior cui propterea non immorandum censeo. Veluti si proponatur haec aequatio:

$$xxddy - xdxdy + ax^n ydx^2 = bx^m dx^2$$

series a potestate x^m est inchoanda, ponendo

$$y = Ax^m + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n} + \text{etc.}$$

vnde fit

$$\begin{array}{lll} m(m-1)Ax^m & +(m+n)(m+n-1)Bx^{m+n} & +(m+2n)(m+2n-1)Cx^{m+2n}\ \text{etc.} \\ -mA & -(m+n)B & -(m+n)C \\ -b & +Aa & +Ba \end{array}$$

hincque

$$A = \frac{b}{m(m-1)};\quad B = \frac{-Aa}{(m+n)(m+n-1)};\quad C = \frac{-Ba}{(m+2n)(m+2n-1)}\ \text{etc.}$$

vbi

vbi quidem multa obſeruanda occurrunt, quae per praecepta ſupra data expedire licet. Imprimis autem in hoc negotio iuuat aequationem propoſitam ope ſubſtitutionis in alias transformaſſe, quarum reſolutio per ſeries fiat ſimplicior, quod cum pluribus modis fieri poſſit, hoc argumentum ſequenti capite diligentius pertractare viſum eſt, ideoque pro forma aequationum

$$ddy + Mdxdy + Nydx^2 = 0$$

quandoquidem pro aliis formis huiusmodi transformatio raro locum inuenit.

 CAPVT IX.

CAPVT IX.

DE TRANSFORMATIONE AEQVATIONVM DIFFERENTIO - DIFFERENTIALIVM

$$Lddy + Mdxdy + Nydx^2 = 0.$$

Problema 125.

993.

Aequationem differentio - differentialem

$$Lddy + Mdxdy + Nydx^2 = 0$$

in qua L, M, N sunt functiones quaecunque ipsius x sumto elemento dx constante, ope substitutionis $y = e^{\int Pdx}z$ in aliam formam transmutare.

Solutio.

Cum hinc sit $\frac{dy}{y} = Pdx + \frac{dz}{z}$, erit differentiando

$$\frac{ddy}{y} - \frac{dy^2}{y^2} = dxdP + \frac{ddz}{z} - \frac{dz^2}{zz} \text{ ergo}$$

$$\frac{ddy}{y} = \frac{ddz}{z} + \frac{2Pdxdz}{z} + dxdP + PPdx^2.$$

Quare

Quare cum aequatio nostra sit

$$\frac{Lddy}{y} + \frac{Mdxdy}{y} + Ndx^2 = 0$$

erit facta substitutione

$$\frac{Lddz}{z} + \frac{2LPdxdz}{z} + LdxdP + LPPdx^2$$
$$+ \frac{Mdxdz}{z} + MPdx^2 + Ndx^2 = 0$$

seu per z multiplicando

$$Lddz + (2LP + M)dxdz + zdx(LdP + LPPdx + MPdx + Ndx) = 0$$

vbi pro P functionem quamcunque ipsius x accipere licet, vnde innumerabiles aequationes inter binas variabiles x et z obtinentur.

Coroll. 1.

994. Quodsi ergo hanc aequationem transformatam integrare vel per seriem resoluere liceat, ex inuento valore ipsius z habebitur $y = e^{\int Pdx} z$.

Coroll. 2.

995. Aequatio transformata similis est propositae, propterea quod in ea variabilis z cum suis differentialibus dz et ddz vbique vnicam dimensionem occupat, perinde ac y in aequatione proposita.

Coroll. 3.

996. Si eueniat, vt ambae aequationes proposita ac transformata aeque commode per series resolui

ſolui poſſint, hoc modo plures reſolutiones eiusdem aequationis exhiberi poſſunt.

Scholion 1.

997. Cum aequationes commode per ſeries reſolubiles in hac forma contineantur:

$$xx(a+bx^n)ddy+x(c+ex^n)dxdy+(f+gx^n)ydx^2=0$$

vbi eſt

$$L=xx(a+bx^n);\ M=x(c+ex^n);\ N=f+gx^n.$$

vt transformata ſimilem obtineat formam, fieri oportet $LP=x(\mu+\nu x^n)$; ideoque $P=\frac{\mu+\nu x^n}{x(a+bx^n)}$. Hinc erit

$$dP=\frac{-\mu a-\mu(n+1)bx^n+\nu(n-1)ax^n-\nu bx^{2n}}{xx(a+bx^n)^2}dx$$

ideoque

$$LdP+LPPdx+MPdx=$$

$$\frac{\begin{matrix}-\mu a-(n+1)\mu bx^n+(n-1)\nu ax^n-\nu bx^{2n}\\ +\mu\mu+2\mu\nu x^n \qquad\qquad +\nu\nu x^{2n}\\ +\mu c+\mu ex^n+\nu cx^n \qquad\qquad +\nu ex^{2n}\end{matrix}}{a+bx^n}$$

vbi diuiſio per $a+bx^n$ ſuccedere debet. Statuatur quotus $=\mu b+\nu kx^n$ fietque

$$\mu=a-c+ab;\ \nu=b-e+bk$$

ac

ac praeterea

$$2\mu\nu-(n+1)\mu b+(n-1)\nu a+\mu e+\nu c=\mu bb+\nu ak$$

vbi priores valores ſubſtituti praebent:

$$(b-k+n)(bc-ae)=nab(b-k) \text{ ſeu } +ab(b-k)^2$$

vnde fit vel

$$b-k=\frac{bc-ae}{ab} \text{ vel } b-k=-n.$$

Litterarum ergo b et k altera arbitrio noſtro relinquitur fitque aequatio transformata

$$xx(a+bx^n)ddz+x(2\mu+c+(2\nu+e)x^n)dxdz$$
$$+(f+\mu b+(g+\nu k)x^n)zdx^2=0.$$

Huius autem reſolutio tam per ſeries aſcendentes, quam deſcendentes ſimiles ipſius x poſtulat poteſtates: Subſtitutio autem ipſa fit

$$y=x^{\frac{a-c}{a}+b}(a+bx^n)^{\frac{bc-ae}{nab}-\frac{(b-k)}{n}}z$$

vbi ne ſola poteſtas ipſius x ingrediatur ſumi debet $b-k=-n$. Nihil intereſt quomodo hic b accipiatur; ſumto ergo $b=0$, fit $k=n$, et ſubſtitutio

$$y=x^{\frac{a-c}{a}}(a+bx^n)^{\frac{bc-ae}{nao}+1}z$$

quae ducit ad hanc aequationem

$$xx(a+bx^n)ddz+x(2a-c+(2(n+1)b-e)x^n)dxdz$$
$$+(f+n(n+1)b-ne+g)x^n)zdx^2=0.$$

Scholion 2.

998. Supra §. 970. vidimus aequationem propositam inter x et y algebraicum admittere integrale, si fuerit

$$\frac{c}{2a}-\frac{e}{2b} \,..\, \frac{\sqrt{((a-c)^2-4af)}}{2a} \,..\, \frac{\sqrt{((b-e)^2-4bg)}}{2b}=in$$

quae si transformata simili modo tractetur, integrale algebraicum assignari poterit, si fuerit

$$-\frac{c}{2a}+\frac{e}{2b} \,..\, \frac{\sqrt{((a-c)^2-4af)}}{2a} \,..\, \frac{\sqrt{(b-e)^2-4bg)}}{2b}-n=in$$

quibus conditionibus coniunctis concludere licet, integrale algebraicum satisfacere, dummodo haec formula

$$\frac{bc-ae}{2ab} \,..\, \frac{\sqrt{((a-c)^2-4af)}}{2a} \,..\, \frac{\sqrt{((b-e)^2-4bg)}}{2b}$$

diuisibilis extiterit per exponentem n, hic signum .. ad ambiguitatem positiui ac negatiui designandam adhibui. Quare si ponamus

$$f=\frac{(a-c)^2-bb}{4a} \text{ et } g=\frac{(b-e)^2-kk}{4b},$$

integrabilitas locum habet, quoties haec expressio $\frac{bc-ae\pm bb\pm ak}{2nab}$ fuerit numerus integer, siue positiuus, siue negatiuus.

Exemplum.

999. *Proposita aequatione*

$$xx(1-xx)ddy+x(1+2mxx)dxdy-m(m+1)xxydx^2=0$$

inuenire casus, quibus integrale algebraicum saltem particulare assignari potest.

Hic

Hic eſt $a=1$, $b=-1$, $c=1$, $e=2m$, $f=0$, $g=-m(m+1)$ et $n=2$. Hinc deducimus

$$h=\sqrt{((a-c)^2-4af)}=0, \text{ et}$$

$$k=\sqrt{((b-e)^2-4bg)}=\sqrt{((2m+1)^2-4m(m+1))}$$

hoc eſt $k=\pm 1$. Formula ergo numero integro aequalis eſt $=\frac{-1-2m\pm 1}{-4}$, vnde geminos pro m caſus nanciſcimur:

$$\text{vel } 2m+2=\pm 4i \text{ vel } 2m=\pm 4i, \text{ hoc eſt}$$

$$\text{vel } m=\pm 2i-1 \text{ vel } m=\pm 2i$$

dummodo ergo m ſit numerus integer ſiue poſitiuus ſiue negatiuus integrale particulare algebraicum exhiberi poteſt. Subſtitutio autem aequationem transformatam praebens eſt

$$y=(1-xx)^{\frac{-1-2m}{-2}+1}z=(1-xx)^{\frac{2m+3}{2}}z$$

ipſa vero aequatio transformata:

$$xx(1-xx)ddz+x(1-2(m+3)xx)dxdz$$
$$-(m+2)(m+3)xxzdx^2=0$$

quam ex illa oriri manifeſtum eſt, ſi loco m ſcribatur $-m-3$. Ipſa autem haec integralia reperiuntur, ob $\lambda\lambda=0$ ponendo

$$y=A+Bx^2+Cx^4+Dx^6+Ex^8+\text{etc.}$$

vnde fit

$$\left.\begin{array}{lllll} 2Bxx & +12Cx^4 & +30Dx^6 & +56Ex^8 \text{ etc.} \\ & -2B & -12C & -30D \\ +2B & +4C & +6D & +8E \\ & +4mB & +8mC & +12mD \\ -m(m+1)A & -m(m+1)B & -m(m+1)C & -m(m+1)D \end{array}\right\} = 0$$

Ergo determinatio coefficientium ita se habet:

$$B = \frac{m(m+1)}{4}A;\quad C = \frac{(m-1)(m-2)}{16}B;\quad D = \frac{(m-3)(m-4)}{36}C \text{ etc.}$$

Ac si ponatur

$$z = \mathfrak{A} + \mathfrak{B}x^2 + \mathfrak{C}x^4 + \mathfrak{D}x^6 + \mathfrak{E}x^8 \text{ etc. erit}$$

$$\mathfrak{B} = \frac{(m+2)(m+3)}{4}\mathfrak{A};\quad \mathfrak{C} = \frac{(m+4)(m+5)}{16}\mathfrak{B};\quad \mathfrak{D} = \frac{(m+5)(m+7)}{36}\mathfrak{C} \text{ etc.}$$

Problema 126.

1000. Aequationem differentio-differentialem

$$Lddy + Mdxdy + Nydx^2 = 0$$

ope substitutionis $\frac{dy}{y} = \frac{Pzdx^2}{dz}$ in aliam eiusdem formae transmutare.

Solutio.

Hic scilicet quaeritur, qualem functionem ipsius x pro P accipi oporteat, vt facta substitutione variabilis z cum suis differentialibus dz et ddz vbique vnicam dimensionem obtineat. Cum igitur sit $\frac{dy}{y} = \frac{Pzdx^2}{dz}$ erit differentiando:

$$\frac{ddy}{y} - \frac{dy^2}{y^2} = \frac{-Pzdx^2ddz}{dz^2} + \frac{zdx^2dP}{dz} + Pdx^2 \text{ et}$$

$$\frac{ddy}{y} = \frac{-Pzdx^2ddz}{dz^2} + \frac{zdx^2dP}{dz} + Pdx^2 + \frac{PPzzdx^4}{dz^2}$$

quibus

quibus valoribus ſubſtitutis fit

$$\frac{-LPzdx^2ddz}{dz^2} + \frac{Lzdx^2dP}{dz} + LPdx^2 + \frac{LPPzzdx^4}{dz^2}$$
$$+ \frac{MPzdx^3}{dz} + Ndx^2 = 0.$$

Sumamus ergo $LP + N = 0$ ſeu $P = \frac{-N}{L}$, et multiplicando per $\frac{-dz^2}{Pzdx^2}$, nanciſcemur:

$$Lddz - \frac{LdPdz}{P} - LPzdx^2 - Mdxdz = 0 \text{ ſeu}$$
$$Lddz - Mdxdz - \frac{LdNdz}{N} + dLdz + Nzdx^2 = 0.$$

Aequatio ergo propoſita ope ſubſtitutionis $\frac{dy}{y} = \frac{-Nzdx^2}{Ldz}$ transformatur in hanc:

$$Lddz + \left(\frac{dL}{dx} - M - \frac{LdN}{Ndx}\right)dxdz + Nzdx^2 = 0.$$

Quodſi ergo hinc valor ipſius z erui poſſit, habebitur etiam valor ipſius y per x expreſſus.

Coroll. 1.

1001. Si in hac aequatione transformata viciſſim ponatur $\frac{dz}{z} = \frac{-Nydx^2}{Ldy}$, ipſa aequatio propoſita exoritur, vnde hae duae aequationes ita inter ſe cohaerent, vt altera ex altera per ſimilem ſubſtitutionem producatur.

Coroll. 2.

1002. Si in aequatione transformata ſecundum ſubſtitutionem priorem ponatur $\frac{dz}{z} = Qdx + \frac{dv}{v}$,

 obti-

obtinebitur haec noua transformata:

$$Lddv+\left(2LQ+\frac{dL}{dx}-M-\frac{LdN}{Ndx}\right)dxdv$$
$$+vdx\left(LdQ+LQQdx+QdL-MQdx-\frac{LQdN}{N}+Ndx\right)=0$$

quae ergo ex ipsa proposita deducitur ponendo $\frac{dy}{y}=\frac{-Nvdx^2}{L(dv+Qvdx)}$.

Scholion 1.

1003. Hinc combinando ambas substitutiones, quibus in binis praecedentibus problematibus sumus vsi, substitutionem huiusmodi generalem adipiscimur:

$$\frac{dy}{y}=\frac{Pdz+Qzdx}{Rdz+Szdx}.dx,$$

quae si in aequatione proposita

$$Lddy+Mdxdy+Nydx^2=0$$

substituatur, functiones P, Q, R, S ita definiri debent, vt in aequatione resultante variabilis z cum suis differentialibus nusquam plus vna dimensione teneat. Oriuntur autem termini quadrato dz^2 affecti, ad quos destruendos fieri oportet:

$$Ldx(PP+QR-PS)+L(RdP-PdR)+MPRdx+NRRdx=0$$

seu $Q=\frac{PS}{R}-\frac{PP}{R}-\frac{dP}{dx}+\frac{PdR}{Rdx}-\frac{MP}{L}-\frac{NR}{L}$

tum vero peruenitur ad hanc aequationem:

$$Lddz(PS-QR)+Ldz(RdQ-QdR+SdP-PdS)+Lzdx(SdQ-QdS+QQdx)$$
$$+dxdz(2LPQ+M(QR+PS)+2NRS)+Szdx^2(MQ+NS)=0.$$

Verum

Verum facilius ad hanc aequationem generalem peruenitur, si ambae substitutiones alternatim in vsum vocentur.

Scholion 2.

1004. Transformatio autem hic exposita eo magis est notatu digna, quod etiamsi aequatio transformata resolutionem admittat, inde tamen non nisi difficulter ipsa proposita resoluatur. Cum enim reperta fuerit functio ipsius x, quae loco z substituta aequationi transformatae satisfaciat, pro valore ipsius y inueniendo, insuper integrale huius aequationis $\frac{dy}{y} = -\frac{Nzdx^2}{Ldz}$ inuestigari oportet, vbi etsi variabiles x et y a se inuicem sunt separatae, tamen difficultates insignes in ipsa integratione se exerere possunt. Fieri ergo poterit, vt ope huius substitutionis, eiusmodi aequationum integralia exhiberi queant, quae directa via vix inuestigare liceat. Scilicet si eueniat, vt integrale aequationis transformatae vel ope methodi cuiusdam supra expositae inueniri, vel per seriem abruptam exprimi possit, tum etiam ipsius aequationis propositae integrale habebitur. Etsi enim casu posteriori integrale tantum particulare innotescit, tamen ex eo semper in hoc aequationum genere integrale completum elici potest. Namque si aequationi

$$Lddy + Mdxdy + Nydx^2 = o$$

par-

particulariter satisfaciat valor $y=X$ ponatur $y=Xv$, fietque

$$\left.\begin{array}{l} LXddv+2LdXdv+LvddX \\ \quad +MXdxdv+MvdxdX \\ \quad\quad +NXvdx^2. \end{array}\right\}=0$$

At quia $X=y$ per hypothesin aequationi satisfacit, erit

$$LddX+MdxdX+NXdx^2=0 \text{ et}$$

$$LXddv+(2LdX+MXdx)dv=0 \text{ seu}$$

$$\frac{ddv}{dv}+\frac{2dX}{X}+\frac{Mdx}{L}=0,$$

vnde integrando oritur

$$XXdv=Ce^{-\int\frac{Mdx}{L}}dx \text{ porroque}$$

$$v=\int\frac{Cdx}{XX}e^{-\int\frac{Mdx}{L}}$$

ita vt integrale completum fit

$$y=CX\int\frac{dx}{XX}e^{-\int\frac{Mdx}{L}}$$

quod ergo ex quolibet integrali particulari $y=X$ elici potest.

Exemplum.

1005. *Aequationem differentio-differentialem:*

$$xx(a+bx^n)ddy+x(c+ex^n)dxdy+fydx^2=0$$

transformare ac per seriem integrare.

Cum

Cum hic sit $L=xx(a+bx^n)$; $M=x(c+ex^n)$ et $N=f$ vtendum est hac substitutione:

$$\frac{dy}{y}=\frac{-fzdx^2}{xx(a+bx^n)dz}$$

qua nostra aequatio reducitur ad hanc formam:

$$xx(a+bx^n)ddz+x(2a-c+((n+2)b-e)x^n)dxdz+fzdx^2=0$$

pro cuius resolutione si ponatur

$$z=Ax^\lambda+Bx^{\lambda+n}+Cx^{\lambda+2n}+\text{ etc.}$$

fieri debet

$$\lambda(\lambda-1)a+\lambda(2a-c)+f=0 \text{ seu}$$

$$\lambda\lambda a+\lambda(a-c)+f=0 \text{ ergo } \lambda=\frac{-a+c\pm\sqrt{((a-c)^2-4af)}}{2a}.$$

Series autem abrumpetur per 970, si haec expressio

$$-\frac{c}{2a}+\frac{e}{2b}-\frac{n}{2}\pm\left(\frac{e}{2b}-\frac{n}{2}-\frac{1}{2}\right)\pm\frac{\sqrt{((a-c)^2-4af)}}{2a}=in$$

denotante i numerum integrum positiuum, hoc est:

$$\text{vel } -\frac{c}{2a}+\frac{e}{b}-\frac{1}{2}-n\pm\frac{\sqrt{((a-c)^2-4af)}}{2a}=in$$

$$\text{vel } -\frac{c}{2a}+\frac{1}{2}\pm\frac{\sqrt{((a-c)^2-4af)}}{2a}=in.$$

Sin autem ipsa aequatio proposita hoc modo in seriem resoluatur, haec abrumpetur, si fuerit

$$\frac{c}{2a}-\frac{e}{2b}\pm\frac{(b-e)}{2b}\pm\frac{\sqrt{((a-c)^2-4af)}}{2a}=in$$

hoc est

$$\text{vel } \frac{c}{2a}-\frac{e}{b}+\frac{1}{2}\pm\frac{\sqrt{((a-c)^2-4af)}}{2a}=in$$

$$\text{vel } \frac{c}{2a}-\frac{1}{2}\pm\frac{\sqrt{((a-e)^2-4af)}}{2a}=in.$$

 Vnde

Vnde intelligitur integrale finitum exhiberi posse, siue numerus integer i sit positiuus siue negatiuus. Ad hanc vero duplicitatem iam prior substitutio perduxerat (998.), ita vt haec noua substitutio nullos nouos casus integrabiles suppeditet.

Scholion 1.

1006. Vt tamen pateat, quomodo ex valore finito ipsius z valor finitus ipsius y elici queat, contemplemur casum

$$xx(a+bx^2)ddy+x(3a+ex^2)dxdy-24aydx^2=0$$

vbi $n=2$, $c=3a$ et $f=-24a$, quae facta substitutione $\frac{dy}{y}=\frac{24azdx^2}{xx(a+bx^2)dz}$ abit in hanc:

$$xx(a+bx^2)ddz+x(-a+(4b-e)xx)dxdz-24azdx^2=0$$

vbi pro serie ascendente fit

$$\lambda\lambda-2\lambda-24=0=(\lambda-6)(\lambda+4)$$

Statuatur

$$z=Ax^{-4}+Bx^{-2}+C+Dx^2 \text{ etc.}$$

erit

$$\left.\begin{array}{llll} 20Aax^{-4}+ & 6Bax^{-2} & * & +2Dx^2 \\ & +20Ab & +6Bb & * \\ +4Aa & +2Ba & * & -2Da \\ & -4A(4b-e) & -2B(4b-e) & * \\ -24Aa & -24Ba & -24Ca & -24Da \end{array}\right\}=0$$

Cum

Cum ergo fit $D=0$, fequentes termini omnes tolluntur. Tum vero eft

$$16Ba=4A(b+e);\; 24Ca=-2Bb+2Be,$$

ergo

$$B=\frac{b+e}{4a}A;\; C=\frac{e-b}{12a}B=\frac{ee-bb}{48aa}A,$$

hincque

$$z=A\left(\frac{1}{x^4}+\frac{b+e}{4ax^2}+\frac{ee-bb}{48aa}\right)=\frac{A(48aa+12a(b+e)xx+(ee-bb)x^4)}{48aax^4}$$

vnde fequitur

$$dz=Adx\left(\frac{-4}{x^5}-\frac{b-e}{2ax^3}\right)=\frac{-Adx}{2ax^5}(8a+(b+e)xx).$$

Ergo

$$\frac{dy}{y}=\frac{-(48aa+12a(b+e)xx+(ee-bb)x^4)}{x(a+bx^2)(8a+(b+e)xx)}dx$$

feu refoluendo

$$\frac{dy}{y}=\frac{-6dx}{x}+\frac{(5b-e)xdx}{a+bxx}+\frac{2(b+e)xdx}{8a+(b+e)xx},$$

hincque integrando

$$y=\frac{A}{x^6}(a+bxx)^{\frac{5b-e}{2b}}(8a+(b+e)xx).$$

Scholion 2.

1007. Quod hic cafu fortuito eueniffe videtur, vt ex valore ipfius z inuento quantitas y commode definiri potuerit, idem perpetuo euenire oportere, fequenti modo in genere oftendi poteft. Cum enim aequatio propofita

$$Lddy+Mdxdy+Nydx^2=0$$

ope substitutionis $\frac{dy}{y} = \frac{-Nzdx^2}{Ldz}$ in hanc fit transformata:

$$Lddz - Mdxdz - \frac{LdNdz}{N} + dLdz + Nzdx^2 = 0$$

si haec per Ldz diuidatur, prodit

$$\frac{ddz}{dz} - \frac{Mdx}{L} - \frac{dN}{N} + \frac{dL}{L} = -\frac{Nzdx^2}{Ldz} = \frac{dy}{y}$$

ex qua integrando elicitur

$$y = \frac{\alpha L dz}{N dx} e^{-\int \frac{Mdx}{L}}$$

quae inuento valore ipsius z statim sine vlteriori integratione praebet valorem ipsius y.

Cum porro sit

$$dy = \frac{-Nyzdx^2}{Ldz} \text{ erit } dy = -\alpha z dx . e^{-\int \frac{Mdx}{L}}$$

hincque

$$ydy = \frac{-\alpha\alpha L z dz}{N} e^{-2\int \frac{Mdx}{L}}.$$

atque hae relationes eo magis sunt notatu dignae, quod ex iis aequatio proposita nonnisi per plures ambages ad transformatam reduci possit. Ipsa enim formula pro y substituta perducit ad aequationem differentialem tertii gradus, quae autem manifesto integrationem admittit, ipsamque aequationem hic inuentam suppeditat. Hinc igitur occasionem adipiscimur eiusmodi substitutiones inuestigandi, quae quidem ad differentialia tertii gradus ascendant, verum tamen per integrationem ad differentialia secunda redigi se patiantur.

Proble-

Problema 127.

1008. Aequationem differentio-differentialem

$$L\,ddy + M\,dx\,dy + N\,y\,dx^2 = 0$$

ope huiusmodi substitutionis $y = \frac{P\,dz}{dx}$ in aliam aequationem pariter differentio-differentialem transformare.

Solutio.

Ob $y = \frac{P\,dz}{dx}$ fit

$$dy = \frac{P\,ddz + dP\,dz}{dx} \quad \text{et} \quad ddy = \frac{P\,d^3z + 2\,dP\,ddz + dz\,ddP}{dx}$$

quibus formulis substitutis oritur haec aequatio differentialis tertii gradus:

$$LP\,d^3z + 2L\,dP\,ddz + L\,dz\,ddP + MP\,dx\,ddz + M\,dx\,dP\,dz + NP\,dx^2\,dz = 0$$

quam ita comparatam assumamus, vt per functionem ipsius x, quae sit Q, integrabilis euadat. Integrabilis ergo sit haec forma:

$$LPQ\,d^3z + 2LQ\,dP\,ddz + MPQ\,dx\,ddz + LQ\,dz\,ddP + MQ\,dx\,dP\,dz + NPQ\,dx^2\,dz = 0$$

cuius integrale sit

$$LPQ\,ddz + S\,dx\,dz + T\,z\,dx^2 = C\,dx^2$$

vnde colligitur

$$ddz(2LQ\,dP + MPQ\,dx) = ddz(d.LPQ + S\,dx)$$
$$dz(LQ\,ddP + MQ\,dx\,dP + NPQ\,dx^2) = dz(dx\,dS + T\,dx^2)$$

et $z\,dx^2\,dT = 0$, ideoque T quantitas constans.

 Inde

Inde autem fit,

$$S dx = LQ dP - LP dQ - PQ dL + MPQ dx$$

ex quo per alteram conditionem elicitur:

$$T dx^2 = LQ ddP + MQ dx dP + NPQ dx^2 - LQ ddP - L dP dQ - Q dP dL$$
$$+ LP ddQ + L dP dQ + P dQ dL + PQ ddL + P dQ dL + Q dP dL$$
$$- MP dx dQ - MQ dx dP - PQ dx dM \text{ siue}$$

$$T dx^2 = P dd.LQ - P dx d.MQ + P.NQ dx^2.$$

Quare cum T sit quantitas constans, ponatur $T = a$ atque hinc commode functio P definitur, scilicet

$$P = \frac{a dx^2}{dd.LQ - dx d.MQ + NQ dx^2}$$

hocque valore pro P assumto, aequatio proposita ope substitutionis $y = \frac{P dz}{dx}$ transformatur in hanc:

$$LPQ ddz + dz(LQ dP - LP dQ - PQ dL + MPQ dx) + az dx^2 = C dx^2$$

vbi cum z constante quantitate augere liceat, constans C omitti potest. Diuidatur ergo haec aequatio per PQ et prodibit:

$$L ddz + dz\left(\frac{L dP}{P} - \frac{L dQ}{Q} - dL + M dx\right) + \frac{az dx^2}{PQ} = 0$$

seu in postremo termino valorem ipsius P substituendo

$$L ddz + dz\left(\frac{L dP}{P} - \frac{d.LQ}{Q} + M dx\right) + \frac{z}{Q}(dd.LQ - dx d.MQ + NQ dx^2) = 0$$

atque hic pro Q functionem quamcunque ipsius x accipere licet.

Coroll. 1.

Coroll. 1.

1009. Hinc praecedens fubftitutio deriuatur ponendo

$$dd.LQ - dxd.MQ = 0 \text{ ideoque } d.LQ - MQdx = Cdx$$

feu $e^{-\int\frac{Mdx}{L}} \cdot LQ = C\int e^{-\int\frac{Mdx}{L}} dx + D.$

Namque fi hic capiatur $C = 0$, erit

$$Q = \frac{D}{L} e^{+\int\frac{Mdx}{L}}, \text{ et } P = \frac{\alpha dx^2}{NQdx^2} = \frac{\alpha}{NQ} \text{ feu}$$

$$P = \frac{\alpha L}{N} e^{-\int\frac{Mdx}{L}}, \text{ vt ante.}$$

Coroll. 2.

1010. Sin autem ponamus

$$dd.LQ - dx\,d.MQ = dXdx \text{ vt fit}$$

$$P = \frac{\alpha dx}{dX + NQdx}, \text{ erit}$$

$$d.LQ - MQdx = Xdx + Adx$$

porroque integrando

$$e^{-\int\frac{Mdx}{L}} LQ = \int e^{-\int\frac{Mdx}{L}} dx(X+A) + B \text{ et}$$

$$Q = \frac{1}{L} e^{\int\frac{Mdx}{L}} \int e^{-\int\frac{Mdx}{L}} dx(X+A) + \frac{B}{L} e^{\int\frac{Mdx}{L}}.$$

Coroll. 3.

1011. Ponatur $\int e^{-\int\frac{Mdx}{L}} Xdx = e^{-\int\frac{Mdx}{L}} V$ et $A = 0$, $B = 0$ erit $X = \frac{dV}{dx} - \frac{MV}{L}$ et $Q = \frac{V}{L}$, ideoque

que $P = \frac{\alpha dx}{\frac{ddV}{dx} - \frac{M}{L} dV - V d. \frac{M}{L} + \frac{N}{L} V dx}$. Si igitur sit $V = \alpha$ erit $Q = \frac{\alpha}{L}$; $P = \frac{LLdx}{LNdx - LdM + MdL}$, aequatioque resultans

$$L ddz + dz \left(\frac{LdP}{P} + M dx \right) + \frac{z dx (LNdx - LdM + MdL)}{L} = 0.$$

Scholion.

1012. Haec autem nimis sunt generalia, quam vt inde quicquam ad vsum communem concludi possit. Vtcunque autem transformatio instituatur, et aequatio transformata in seriem resoluatur, haec nullis aliis casibus abrumpi videtur, nisi iis quibus ipsa aequatio proposita, et inde per primam substitutionem transformata, ad seriem alicubi terminatam reducitur. Ex quo perspicuum est ope huiusmodi transformationum vix vnquam nouos casus integrabiles erui posse. Verum dum hactenus loco variabilis y aliam z per substitutionem introduximus, altera x, ex cuius potestatibus series formabantur, retenta, nunc etiam paucis exploremus, quomodo loco ipsius x aliam variabilem t introducendo, transformationem perfici oporteat, vbi imprimis notetur necesse est, cum ante elementum dx assumtum fuerit constans, iam in transformata elementum dt constans accipi debere. Hic igitur t scribetur loco certae cuiuspiam functionis ipsius x, quam autem ita comparatam esse decet, vt aequatio resultans ne nimis fiat complicata.

Proble-

Problema 128.

1013. Propofita aequatione differentio-differentiali

$$Lddy + Mdxdy + Nydx^2 = 0.$$

Loco quantitatis x aliam t introducere, quae functioni cuipiam ipfius x aequetur.

Solutio.

Diuifa aequatione per dx, repraefentetur aequatio ita

$$Ld.\frac{dy}{dx} + Mdy + Nydx = 0$$

vt iam confideratio elementi dx, quod conftans erat affumtum, fit exclufa. Cum t aequetur functioni cuipiam ipfius x, fiat inde $dt = Pdx$, feu $dx = \frac{dt}{P}$ vnde nacifcimur

$$Ld.\frac{Pdy}{dt} + Mdy + \frac{Nydt}{P} = 0$$

ac fumto elemento dt conftante:

$$LPddy + LdPdy + Mdtdy + \frac{Nydt^2}{P} = 0$$

vbi tantum fupereft, vt in quantitatibus finitis, quae adhuc variabilem x complectuntur, eius loco altera t introducatur.

Exemplum.

1014. *Propofita fit haec aequatio:*

$$xx(a+bx^n)ddy + x(c+ex^n)dxdy + (f+gx^n)ydx^2 = 0$$

in quam loco formulae $h+kx^n$ *introducatur* t.

 Cum

Cum ergo sit

$$t = b + kx^n \text{ erit } dt = nkx^{n-1}dx,$$

ideoque

$$P = nkx^{n-1} \text{ et } dP = n(n-1)kx^{n-2}dx = \frac{(n-1)dt}{x}$$

Quare habebimus:

$$nkx^{n+1}(a+bx^n)ddy + (n-1)xdtdy(a+bx^n) + x(c+ex^n)dtdy + \frac{(f+gx^n)ydt^2}{nkx^{n-1}} = 0$$

siue

$$nk(a+bx^n)ddy + \frac{(n-1)dtdy(a+bx^n) + dtdy(c+ex^n)}{x^n} + \frac{(f+gx^n)ydt^2}{nkx^{2n}} = 0.$$

Nunc vero est $x^n = \frac{t-b}{k}$, qui valor substitutus praebet:

$$n(ak-bb+bt)ddy + \frac{(n-1)dtdy(ak-bb+bt) + dtdy(ck-eb+et)}{t-b} + \frac{(fk-gb+gt)ydt^2}{n(t-b)^2} = 0$$

Verum hic ita vbique $t-b$ occurrit, vt aequatio simplicior euadat loco $t-b$ scribendo u, tum autem perinde est, ac si loco potestatis x^n scripsissemus quantitatem u: neque ergo hinc quicquam lucri pro nouis seriebus eruendis redundat.

Corol-

Corollarium.

1015. Si in aequatione generali loco x^m scribere velimus t erit

$$dt = mx^{m-1}dx \text{ et } P = mx^{m-1},$$

et aequatio resultabit ob

$$dP = m(m-1)x^{m-2}dx = \frac{(m-1)dt}{x},$$

ista

$$mLx^{m-1}ddy + \frac{(m-1)Ldtdy}{x} + Mdtdy + \frac{Nydt^2}{mx^{m-1}} = 0$$

seu

$$mLddy + \frac{(m-1)Ldtdy}{t} + \frac{Mxdtdy}{t} + \frac{Nxxydt^2}{mtt} = 0.$$

Scholion.

1016. Plura de huiusmodi aequationum transformationibus tradere haud necesse videtur, cum ex his fontibus haud difficulter omnes transformationes ad vsum idoneae deriuari queant. Datur autem alia methodus prorsus singularis huiusmodi aequationum differentio-differentialium integralia exprimendi, quae per formulas integrales binas variabiles inuoluentes expeditur, dum altera in integratione vt constans tractatur. Ita si P fuerit functio quaecunque binarum variabilium x et u, ac ponatur $y = \int Pdx$, considerando u in integratione vt constantem, integrale hoc $\int Pdx$ erit functio ipsarum x et u, quod

 ita

ita determinatum vt euanefcat, pofito $x=0$, fi deinceps ftatuatur $x=a$, obtinebitur functio ipfius u ipfi y aequalis, quae fi fatisfaciat aequationi cuipiam differentiali inter u et y propofitae, haec aequatio refoluetur formula $y=\int P\,dx$, quae vt eius integrale fpectari poterit. Atque hoc modo innumerabilium aequationum differentia-differentialium integralia exhiberi poffunt, quae aliis methodis prorfus intractabiles videntur. Quanquam autem formula $\int P\,dx$, fpectata quantitate u vt conftante, actu integrari nequit, tamen eius integrale in hoc negotio pro cognito accipi poteft, quia eius valor faltem per approximationes affignari poteft. Scilicet dum fumta x pro abfciffa, fi P denotet applicatam orthogonalem ei conuenientem, formula $\int P\,dx$ exprimet aream eiusdem curuae abciffae x infiftentem, ac pofito $x=a$, area habetur determinata valori $y=\int P\,dx$, prouti cum modo definiuimus, aequalis, quae ergo, vti loqui folent, per quadraturas curuarum affignari poteft, ex quo haec integrandi ratio commode appellatur conftructio per quadraturas. Hic autem imprimis ad eam rationem, qua integralia in particularia et completa diftinximus, attendi conueniet; vnde follicite eft cauendum, ne integralia hoc modo inuenta pro completis habeantur, nifi quatenus binas conftantes arbitrarias inuoluant. Cum igitur eidem aequationi differentiali infinita integralia particularia conueniat, mirandum non eft, fi hoc modo pro eadem aequatione propofita plura integralia diuerfa inue-

inueniamus. Hoc autem argumentum fere prorſus eſt nouum, neque a quoquam adhuc pertractatum, ſiquidem nonnulla ſpecimina, quae equidem iam dudum dedi, excipiantur; ex quo dubitare non licet, quin iſta methodus, ſi diligentius excolatur, aliquando forte praeclara incrementa in Analyſin ſit allatura.

CAPVT X.

DE CONSTRVCTIONE AEQVATIONVM DIFFERENTIO-DIFFERENTIALIVM PER QVADRATVRAS CVRVARVM.

Problema 129.

1017.

Si fuerit $y=\int V\,dx$ denotante V functionem quamcunque binarum quantitatum x et u, quarum autem haec u in integratione vt constans spectatur, post integrationem vero statuatur $x=a$, vt y aequetur functioni cuidam ipsius u; quodsi iam u variabilis sumatur, inuestigare valorem ipsius $\frac{dy}{du}$.

Solutio.

Cum $\int V\,dx$ exhibeat functionem quandam binarum quantitatum x et u, cuius differentiale sumta u constante est $=V\,dx$, si tam u quam x vt variabiles tractentur, differentiale aequationis $y=\int V\,dx$ talem habebit formam: $dy=V\,dx+U\,du$, quae quia est differentiale verum, necesse est sit $(\frac{dV}{du})=(\frac{dU}{dx})$ At cum V sit functio data ipsarum x et u, ponatur $dV=P\,dx+Q\,du$, eritque $(\frac{dV}{du})=Q$, ideoque $(\frac{dU}{dx})=Q$. Hinc considerata iterum u vt constante, erit

erit $dU = Qdx$, et $U = \int Qdx$, in qua integratione ſola x pro variabili habetur. Quocirca ſi hunc valorem $\int Qdx$ vt cognitum ſpectemus, quippe quem per quadraturas aſſignare licet, erit $dy = Vdx + du \int Qdx$. Quaerimus autem id ipſius y differentiale, quod ex variabilitate ipſius u tantum naſcitur; quod cum ſit $dy = du \int Qdx$ erit valor quaeſitus $\frac{dy}{du} = \int Qdx$, ſi nempe poſt integrationem itidem ponatur $x = a$.

Coroll. 1.

1018. Cum ſit $y = \int Vdx$ functio ipſarum x et u, per integrationem autem formulae Vdx, in qua u conſtans ſpectatur, functio quaecunque ipſius u loco conſtantis accedere poſſit, functio y per ſe erit indeterminata, determinabitur autem ſtatim, atque integrale $\int Vdx$ ita accipiatur, vt euaneſcat poſito $x = 0$.

Coroll. 2.

1019. Hac conditione obſeruata euaneſcet y poſito $x = 0$ quicunque valor alteri quantitati u tribuatur, erit ergo etiam $y + du(\frac{dy}{du}) = 0$ facto $x = 0$, ergo etiam $(\frac{dy}{du}) = 0$. Vnde patet $\int Qdx = \frac{dy}{du}$ ita quoque accipi debere, vt poſito $x = 0$ euaneſcat.

Coroll. 3.

Coroll. 3.

1020. Cnm $y=\int V dx$ erit $(\frac{dy}{dx})=V$, hinc $(\frac{ddy}{du\,dx})=(\frac{dV}{du})$. At fi ponatur $(\frac{dy}{du})=Z$ erit quoque $(\frac{ddy}{du\,dx})=(\frac{dZ}{dx})$ ergo $(\frac{dZ}{dx})=(\frac{dV}{du})$. Quare fpectata u vt conftante erit $dZ=dx(\frac{dV}{du})$, et $Z=\int dx(\frac{dV}{du})$ ideoque $(\frac{dy}{du})=\int dx(\frac{dV}{du})$.

Coroll. 4.

1021. Quodfi ergo poft integrationes ita abfolutas, vt integralia euanefcant pofito $x=0$, ponatur $x=a$, tam valor $y=\int V dx$ quam $\frac{dy}{du}=\int dx(\frac{dV}{du})$ erit functio determinata ipfius u.

Coroll. 5.

1022. Simili modo vlterius progrediendo erit $\frac{ddy}{du^2}=\int dx(\frac{ddV}{du^2})$. Quare fi L, M et N denotent functiones quascunque ipfius u erit

$$\frac{Lddy}{du^2}+\frac{Mdy}{du}+Ny=\int dx(L(\frac{ddV}{du^2})+M(\frac{dV}{du})+NV)$$

totumque negotium huc redit, vt ifta formula integrationem admittat.

Scholion.

1023. Datis fcilicet ipfius u functionibus L, M, N, quaeri debet functio V binarum variabilium x et u, ita vt fpectata u conftante formula

$$(L(\frac{ddV}{du^2})+M(\frac{dV}{du})+NV)dx$$

abfo-

absolute fiat integrabilis, cuius integrale, vt sit determinatum ita capiatur, vt posito $x=0$ euanescat. Tum vero statuatur $x=a$, ac si illud integrale etiam hoc casu euanescat, erit

$$\frac{L\,ddy}{du^2} + \frac{M\,dy}{du} + Ny = 0,$$

hincque aequationi satisfacit valor $y=\int V dx$, lege indicata sumtus. Problema autem datis functionibus L, M et N, inuestigandi functionem V maxime est indeterminatum, neque methodis adhuc cognitis in genere resolui potest; ex quo conueniet id inuerso modo tractari, vt sumta functione V alterae L, M et N indagentur. Hinc aequationes differentio-differentiales consequemur, quarum integralia hoc modo assignare valebimus, quae si aliis methodis tractari nequeant, insigne lucrum suppeditant. Quodsi integrale illud

$$\int \left(L\left(\frac{ddV}{du^2}\right) + M\left(\frac{dV}{du}\right) + NV \right) dx$$

posito $x=a$ non euanescat, sed datam ipsius u functionem U exhibeat, valor $y=\int V\,dx$ conueniet huic aequationi,

$$\frac{L\,ddy}{du^2} + \frac{M\,dy}{du} + Ny = U,$$

quae cum infinitis modis in alias formas transmutari possit, etiam harum integralia innotescent, vbi simul hoc commode euenit, vt etiamsi integrale tantum particulare obtineatur, inde tamen plerumque integrale completum haud difficulter colligi queat.

Problema 130.

1024. Inuenire aequationes differentio - differentiales formae $\frac{L\,ddy}{du^2} + \frac{M\,dy}{du} + Ny = U$, vt L, M, N et U sint functiones ipsius u cuius elementum du hic pro constante accipitur, quarum integrale ope constructionis per quadraturas exhiberi possit.

Solutio.

Sumatur functio quaecunque binarum variabilium u et x, quae sit V, capiaturque integrale $\int V dx$ spectata quantitate u vt constante, ita vt posito $x=0$, euanescat, tum vero fiat $x=a$, denotante a quantitatem quamcunque constantem, vt iam $\int V dx$ exprimat functionem quandam ipsius u tantum cui quantitas y aequetur, vt sit $y=\int V dx$. Cum iam sit

$$\frac{dy}{du} = \int dx\left(\frac{dV}{du}\right), \text{ et } \frac{ddy}{du^2} = \int dx\left(\frac{ddV}{du^2}\right),$$

his integralibus pariter ita sumtis, vt posito $x=0$ euanescant, tum vero statuatur $x=a$, quaerantur functiones L, M, N ipsius u vt haec formula

$$\int dx\left(L\left(\frac{ddV}{du^2}\right) + M\left(\frac{dV}{du}\right) + NV\right)$$

fiat absolute integrabilis, eiusque integrale ita determinetur, vt posito $x=a$, fiat id $=U$. Quod si fuerit praestitum, euidens est aequationi differentio-differentiali

$$\frac{L\,ddy}{du^2} + \frac{M\,dy}{du} + Ny = U$$

satisfacere formulam assumtam $y=\int V dx$.

Coroll. 1.

Coroll. 1.

1025. Assumtio ergo functionis V non penitus arbitrio nostro permittitur, sed ad hoc potissimum est spectandum, vt talis forma

$$\int dx\left(L\left(\frac{ddV}{du^2}\right)+M\left(\frac{dV}{du}\right)+NV\right)$$

per se fiat integrabilis.

Coroll. 2.

1026. Infinitae ergo hinc statim excluduntur formae ad hunc scopum ineptae, cuiusmodi sunt $V=UP$ existente U functione ipsius u et P ipsius x tantum; quia tum foret

$$y=U\int Pdx,\ \frac{dy}{du}=\frac{dU}{du}\int Pdx \text{ et } \frac{ddy}{du^2}=\frac{ddU}{du^2}\int Pdx$$

quippe quae idem integrale complectuntur, ita vt ex earum coniunctione formula absolute integrabilis confici nequeat.

Exemplum 1.

1027. *Sit* $V=x^n\sqrt{\frac{uu+xx}{cc-xx}}$ *et* $y=\int x^n dx\sqrt{\frac{uu+xx}{cc-xx}}$ *integrali euanescente posito* $x=0$, *tum vero facto* $x=a$.

Erit ergo

$$\left(\frac{dV}{du}\right)=x^n\cdot\frac{u}{\sqrt{(uu+xx)(cc-xx)}} \text{ et } \left(\frac{ddV}{du^2}\right)=x^n\cdot\frac{xx}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}$$

et integrabilis reddi debet haec formula

$$x^n dx\left(\frac{Lxx}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}+\frac{Mu}{\sqrt{(uu+xx)(cc-xx)}}+N\sqrt{\frac{uu+xx}{cc-xx}}\right)$$

seu

$$\frac{x^n dx}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}(Lxx+Mu(uu+xx)+N(uu+xx)^2).$$

Statuatur integrale $=\frac{x^{n+1}\sqrt{(cc-xx)}}{\sqrt{(uu+xx)}}$, cuius differentiale cum sit

$$\frac{(n+1)x^n(cc-xx)(uu+xx)-x^{n+2}(uu+xx)-x^{n+2}(cc-xx)}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}\cdot dx$$

seu

$$\frac{x^n dx}{(uu+xx)^{\frac{3}{2}}\sqrt{(cc-xx)}}\left\{\begin{matrix}(n+1)ccuu+(n+1)ccxx-(n+1)uuxx-(n+1)x^4\\ \quad\quad -ccxx \quad\quad -uuxx\end{matrix}\right\}$$

cum qua si proposita comparetur, fiet:

$$Mu^3+Nu^4=(n+1)ccuu$$
$$L+Mu+2Nuu=ncc-(n+2)uu \text{ et}$$
$$N=-(n+1).$$

Hinc elilicitur

$Mu=(n+1)(cc+uu)$ seu $M=\frac{(n+1)(cc+uu)}{u}$ et
$L=-(n+1)(cc+uu)+2(n+1)uu+ncc-(n+2)uu$
seu $L=-cc-uu$.

Quam-

Quamobrem habebimus

$$-\frac{(cc+uu)ddy}{du^2}+\frac{(n+1)(cc+uu)dy}{udu}-(n+1)y=\frac{a^{n+1}\sqrt{(cc-aa)}}{\sqrt{(aa+uu)}}$$

cui aequationi satisfacit $y=\int x^n dx\sqrt{\frac{uu+xx}{cc-xx}}$ integratione absoluta vt est indicatum.

Coroll. 1.

1028. Sumto ergo $a=c$; formula integralis $y=\int x^n dx\sqrt{\frac{uu+xx}{cc-xx}}$ posito post integrationem $x=c$ exhibebit integrale huius aequationis

$$u(cc+uu)ddy-(n+1)(cc+uu)dudy+(n+1)uydu^2=0$$

seu $ddy-\frac{(n+1)dudy}{u}+\frac{(n+1)ydu^2}{cc+uu}=0$.

Coroll. 2.

1029. Si sit $n=1$ per integrationem inuenitur

$$\int xdx\sqrt{\frac{uu+xx}{cc-xx}}=\tfrac{1}{4}(cc+uu)\text{ Ang. sin. }\frac{2xx-cc+uu}{cc+uu}$$
$$-\tfrac{1}{2}\sqrt{(ccuu+ccxx-uuxx-x^4)}$$
$$-\tfrac{1}{4}(cc+uu)\text{ Ang. sin. }\frac{-cc+uu}{cc+uu}+\tfrac{1}{2}cu\,,$$

et posito $x=c$ fit

$y=\tfrac{1}{4}(cc+uu)\text{ Ang. cos. }\frac{uu-cc}{cc+uu}+\tfrac{1}{2}cu$ hincque

$\frac{dy}{du}=\tfrac{1}{2}u\text{ Ang. cos. }\frac{uu-cc}{cc+uu}$ et $\frac{ddy}{du^2}=\tfrac{1}{2}\text{ Ang. cos. }\frac{uu-cc}{cc+uu}-\frac{cu}{cc+uu}$

quae formulae euidenter satisfaciunt aequationi

$$ddy-\frac{2dudy}{u}+\frac{2ydu^2}{cc+uu}=0.$$

 Coroll. 3.

Coroll. 3.

1030. Hoc casu integrale etiam hoc modo exprimi potest

$$y = \tfrac{1}{4}(cc+uu)\,\text{Ang. sin.}\,\frac{2cu}{cc+uu} + \tfrac{1}{2}cu$$

seu cum eius multiplum quoduis aeque satisfaciat:

$$y = (cc+uu)\,\text{Ang. sin.}\,\frac{2cu}{cc+uu} + 2cu$$

satisfacit vero etiam $y = cc + uu$, vnde integrale completum est.

$$y = \alpha(cc+uu)\,\text{Ang. sin.}\,\frac{2cu}{cc+uu} + 2\alpha cu + \beta(cc+uu).$$

Scholion.

1031. Quod valor $y = cc + uu$ satisfaciat, ex integrali inuento concludere licet, quia enim $\text{Ang. sin.}\,\frac{2cu}{cc+uu}$ est functio multiplex et termino 2π augeri potest, integrale ipsium augeri potest termino $2\pi(cc+uu)$. At in genere differentia binorum integralium quoque satisfacit, ergo etiam satisfacere debet $y = 2\pi(cc+uu)$ et generatim $y = \beta(cc+uu)$. Ex hoc casu facilius perspicitur, quomodo valor assumtus aequationi generali satisfaciat, etiam is per integrationem euolui nequeat. Patet autem $n+1$ esse debere numerum positiuum, quia alioquin conditio integralis, vt posito $x = 0$ euanescat, impleri nequit.

Exem-

Exemplum 2.

1032. *Sumatur*

$$V = x^{n-1}(uu+xx)^{\mu}(cc-xx)^{\nu}, \text{ erit}$$

$$\left(\frac{dV}{du}\right) = 2\mu u x^{n-1}(uu+xx)^{\mu-1}(cc-xx)^{\nu} \text{ et}$$

$$\left(\frac{ddV}{du^2}\right) = 2\mu x^{n-1}(cc-xx)^{\nu}((uu+xx)^{\mu-1} + 2(\mu-1)uu(uu+xx)^{\mu-2})$$
$$= 2\mu x^{n-1}(cc-xx)^{\nu}(uu+xx)^{\mu-2}((2\mu-1)uu+xx).$$

Integrabilis igitur reddi debet absolute haec formula

$$\int x^{n-1}dx(cc-xx)^{\nu}(uu+xx)^{\mu-2} \times$$
$$(2\mu((2\mu-1)uu+xx)L + 2\mu u(uu+xx)M + (uu+xx)^2 N) \text{ seu}$$

$$\int x^{n-1}dx(cc-xx)^{\nu}(uu+xx)^{\mu-2}\left\{\begin{matrix} 2\mu(2\mu-1)Luu & +2\mu Lxx & +Nx^4 \\ +2\mu Mu^3 & +2\mu Muxx & \\ +Nu^4 & +2Nuuxx & \end{matrix}\right\}$$

Statuatur integrale $x^n(uu+xx)^{\mu-1}(cc-xx)^{\nu+1}$ cuius differentiale cum sit

$$x^{n-1}dx(uu+xx)^{\mu-2}(cc-xx)^{\nu} \times$$
$$(n(uu+xx)(cc-xx) + 2(\mu-1)xx(cc-xx) - 2(\nu+1)xx(uu+xx))$$

erit

$$2\mu(2\mu-1)Luu + 2\mu Mu^3 + Nu^4 = nccuu$$
$$2\mu L + 2\mu Mu + 2Nuu = ncc - nuu + 2(\mu-1)cc - 2(\nu+1)uu$$
$$N = -n - 2(\mu-1) - 2(\nu+1) = -n - 2\mu - 2\nu.$$

At prima

$$2\mu(2\mu-1)L + 2\mu Mu + Nuu = ncc$$

demta

demta fecunda dat

$$4\mu(\mu-1)L - Nuu = (n+2\nu+2)uu - 2(\mu-1)cc \text{ feu}$$

$$4\mu(\mu-1)L = -2(\mu-1)(uu+cc) \text{ hinc } L = \frac{-cc-uu}{2\mu}$$

qui valor in prima fubftitutus dat:

$$-(2\mu-1)(cc+uu)+2\mu Mu-(n+2\mu+2\nu)uu = ncc$$

feu

$$2\mu Mu = (n+2\mu-1)cc+(n+4\mu+2\nu+1)uu.$$

Ergo $M = \frac{(n+2\mu-1)(cc+uu)}{2\mu u} + \frac{(\mu+\nu)u}{\mu}$

Si $n > 0$ fuperius integrale euanefcit pofito $x=0$, quare fi ponamus $x=a$, orietur haec aequatio:

$$-\frac{(cc+uu)ddy}{2\mu du^2} + \frac{(n-2\mu-1)(cc+uu)dy}{2\mu u du} + \frac{(\mu+\nu)udy}{\mu du}$$
$$-(n+2\mu+2\nu)y = a^n(aa+uu)^{\mu-1}(cc-aa)^{\nu+1}$$

cuius integrale eft

$$y = \int x^{n-1}dx(uu+xx)^\mu(cc-xx)^\nu,$$

integrali hoc ita fumto vt euanefcat pofito $x=0$, tum vero pofito $x=a$.

Coroll. 1.

1033. Si capiatur $a=c$, vt poftrema pars fiat $=0$, fiquidem exponens $\nu+1$ fit nihilo maior, formula $y=\int x^{n-1}dx(uu+xx)^\mu(cc-xx)^\nu$ pofito $x=c$ poft integrationem ita peractam vt cafu $x=0$ fiat $y=0$, erit integrale huius aequationis

$$u(cc+uu)ddy-(n+2\mu-1)(cc+uu)dudy-2(\mu+\nu)uudud y$$
$$+2\mu(n+2\mu+2\nu)uydu^2=0.$$

Coroll. 2.

Coroll. 2.

1034. Sit $n+2\mu-1=\alpha$; $n+4\mu+2\nu-1=\beta$; fiet $2\mu=\alpha+1-n$ et $2\nu=\beta+1-n-2\alpha-2+2n.\ \beta-1+n-2\alpha$ et aequationis

$$u(cc+uu)ddy-(\alpha cc+\beta uu)dudy+(\alpha+1-n)(\beta-\alpha+n)uydu^2=0$$

integrale erit

$$y=\int x^{n-1}dx(uu+xx)^{\frac{\alpha+1-n}{2}}(cc+xx)^{\frac{\beta-1+2n-2\alpha}{2}}$$

posito $x=c$, si modo sit $n>0$ et $\beta+1+n>2\alpha$.

Scholion.

1035. Haec constructio latissime ad hanc aequationem patet:

$$xx(a+bx^n)ddz+x(c+ex^n)dxdz+(f+gx^n)zdx^2=0$$

primo enim hic sine detrimento amplitudinis sumi potest $n=2$, ponendo $x^n=uu$. Tum vero vti supra §. 998. vidimus ponendo

$$z=x^{\frac{a-c}{a}+b}(a+bx)^{\frac{bc-ae}{nab}+1}y$$

aequatio abit in hanc:

$$xx(a+bx^n)ddy+x(2a-c+2ab+(2b-e+2nb+2bb)x^n)dxdy$$
$$+(f+ab-cb+abb+(g+(b-e+nb+bb)(n+b))x^n)ydx^2=0$$

vbi si b ita accipiatur, vt sit $abb+(a-c)b+f=0$ prodit aequatio formae, cuius constructionem dedi-

 mus.

mus. In casibus autem specialibus difficultates occurrere possunt, quibus superandis sequentia exempla inseruiunt.

Exemplum 3.

1036. *Sit* $V = e^{mux} x^n (c-x)^\nu$, *erit* $\left(\frac{dV}{du}\right) = m e^{mux} x^{n+1} (c-x)^\nu$ *et* $\left(\frac{ddV}{du^2}\right) = mm e^{mux} x^{n+2} (c-x)^\nu$.

Integrabilem ergo reddi oportet hanc formulam:

$$e^{mux} x^n dx (c-x)^\nu (mmLxx + mMx + N)$$

cuius integrale ponatur $= e^{mux} x^{n+1} (c-x)^{\nu+1}$, cuius propterea differentiale illi formulae aequari debet: quod cum sit

$$e^{mux} x^n dx (c-x)^\nu (mux(c-x) + (n+1)(c-x) - (\nu+1)x)$$

erit

$$N = (n+1)c;\; mM = mcu - (n+\nu+2);\; mmL = -mu.$$

Statuatur nunc $x = a$, et formula $y = \int e^{mux} x^n dx (c-x)^\nu$ erit integrale huius aequationis

$$-\frac{uddy}{mdu^2} + \frac{cudy}{du} - \frac{(n+\nu+2)dy}{mdu} + (n+1)cy = e^{mau} a^{n+1} (c-a)^{\nu+1}.$$

Hic poni potest $m = 1$, ac sumto $c = a$, aequationis

$$uddy - aududy + (n+\nu+2)dudy - (n+1)aydu^2 = 0$$

integrale est $y = \int e^{ux} x^n dx (a-x)^\nu$, posito post integrationem $x = a$, dum sit $\nu + 1 > 0$, et $n + 1 > 0$ vt integrale euanescens reddi possit posito $x = 0$.

Coroll. 1.

Coroll. 1.

1037. Si hic ponatur $y = e^{\int z\,du}$ erit

$$udz + uzzdu - auzdu + (n+\nu+2)zdu - (n+1)adu = 0$$

cuius integrale est

$$z = \frac{dy}{ydu} = \frac{\int e^{ux} x^{n+1} dx (a-x)^{\nu}}{\int e^{ux} x^{n} dx (a-x)^{\nu}}.$$

Illa aequatio autem posito $z = \frac{1}{2}a + v$, transmutatur in hanc

$$udv + uvvdu + (n+\nu+2)vdu - \tfrac{1}{4}aaudu - \tfrac{1}{2}(n-\nu)adu = 0$$

quae ponendo $v = u^{-n-\nu-2}s$ abit in hanc

$$u^{-n-\nu-1}ds + u^{-2n-2\nu-3}ssdu - \tfrac{1}{4}aaudu - \tfrac{1}{2}(n-\nu)adu = 0.$$

Coroll. 2.

1038. Sit porro

$$u^{-n-\nu-2}du = dt \text{ seu } u^{-n-\nu-1} = -(n+\nu+1)t$$

vt fiat

$$ds + ssdt - \tfrac{1}{4}aau^{2n+2\nu+4}dt - \tfrac{1}{2}(n-\nu)u^{2n+2\nu+3}dt = 0$$

quae ergo aequatio etiam construi potest. Vel sit $-(n+\nu+1)t = r$ erit

$$ds - \frac{ssdr}{n+\nu+1} + \frac{aar^{\frac{-2n-2\nu-4}{n+\nu+1}}dr}{4(n+\nu+1)} + \frac{(n-\nu)r^{\frac{-2n-2\nu-3}{n+\nu+1}}dr}{2(n+\nu+1)} = 0$$

quae posito $s = -(n+\nu+1)q$ abit in:

$$dq + qq\,dr - \frac{aar^{\frac{-2n-2\nu+4}{n+\nu+1}}dr - 2(n-\nu)r^{\frac{-2n-2\nu-1}{n+\nu+1}}dr}{4(n+\nu+1)^2} = 0$$

hicque est

$$u = r^{\frac{-1}{n+\nu+1}} \text{ et } z = \tfrac{1}{2}a - (n+\nu+1)r^{\frac{n+\nu+2}{n+\nu+1}}q.$$

Scholion.

1039. Cum aequationis differentio-differentialis

$$\frac{ddy}{du} - a\,dy + \frac{(n+\nu+2)dy}{u} - \frac{(n+1)ay\,du}{u} = 0$$

integrale sit $y = \int e^{ux}x^n dx(a-x)^\nu$, videamus, quomodo haec ipsa aequatio in alias formas transfundi possit. Sit primo $u = \alpha t^\lambda$ ideoque $du = \alpha\lambda t^{\lambda-1}dt$, vnde fit:

$$\frac{1}{\alpha\lambda}d.\frac{dy}{t^{\lambda-1}dt} - a\,dy + \frac{(n+\nu+2)dy}{\alpha t^\lambda} - \frac{\lambda(n+1)ay\,dt}{t} = 0$$

sumatur iam elementum dt constans, eritque

$$\frac{ddy}{\alpha\lambda t^{\lambda-1}dt} - \frac{(\lambda-1)dy}{\alpha\lambda t^\lambda} - a\,dy + \frac{(n+\nu+2)dy}{\alpha t^\lambda} - \frac{\lambda(n+1)ay\,dt}{t} = 0$$

seu

$$ddy - \alpha\lambda a t^{\lambda-1}dt\,dy + \frac{(\lambda n+\lambda\nu+\lambda+1)dt\,dy}{t} - \alpha\lambda\lambda(n+1)at^{\lambda-2}y\,dt^2 = 0$$

cuius

cuius integrale eſt

$$y = \int e^{\alpha t^\lambda x} x^n dx (a-x)^\nu.$$

Ponatur porro

$$\frac{dy}{y} = P dt + \frac{dz}{z} \text{ vt ſit } z = e^{-\int P dt} y \text{, erit}$$

$$ddz + 2P dt dz - \alpha\lambda a t^{\lambda-1} dt dz + (\lambda n + \lambda\nu + \lambda + 1)\frac{dt dz}{t} + z dt dP$$
$$+ z dt^2 \left(PP - \alpha\lambda a t^{\lambda-1} P + \frac{(\lambda n + \lambda\nu + \lambda + 1)P}{t} - \alpha\lambda\lambda(n+1) a t^{\lambda-2}\right) = 0.$$

Ad terminos elemento dz adfectos tollendos ſtatuatur

$$P = \tfrac{1}{2}\alpha\lambda a t^{\lambda-1} - \frac{(\lambda n + \lambda\nu + \lambda + 1)}{2t} \text{ fietque}$$

ac prodibit haec aequatio:

$$ddz - z dt^2 \left(\frac{(\lambda n + \lambda\nu + \lambda)^2 - 1}{4tt} + \tfrac{1}{2}\alpha\lambda\lambda(n-\nu) a t^{\lambda-2} + \tfrac{1}{4}\alpha^2\lambda^2 a^2 t^{2\lambda-2}\right) = 0$$

cuius propterea integrale eſt;

$$z = e^{-\frac{1}{2}\alpha a t^\lambda} t^{\frac{\lambda n + \lambda\nu + \lambda + 1}{2}} \int e^{\alpha t^\lambda x} x^n dx (a-x)^\nu.$$

Quodſi ergo ſit $\nu = n$; $\lambda\lambda(2n+1)^2 - 1 = 0$; ſeu $\lambda = \frac{\pm 1}{2n+1}$ et $\alpha = \pm\frac{2}{\lambda} = \pm 2(2n+1)$ habebitur haec aequatio

$$ddz - aazt^{\frac{\pm 2}{2n+1} - 2} dt^2 = 0$$

cuius integrale eſt

$$z = e^{\pm(2n+1) a t^{\frac{\pm 1}{2n+1}}} t^{\pm\frac{1}{2} + \frac{1}{2}} \int e^{\pm 2(2n+1) t^{\frac{\pm 1}{2n+1}} x} x^n dx (a-x)^n.$$

Vel huius aequationis

$$ddz - aat^{2\lambda-2} z dt^2 = 0$$

 inte-

integrale eſt

$$z = e^{-\frac{a}{\lambda}t^{\lambda}}\, t \int e^{\frac{2a}{\lambda}t^{\lambda}}\, x^{\pm\frac{1}{2\lambda}-\frac{1}{2}}\, dx (a-x)^{\pm\frac{1}{2\lambda}-\frac{1}{2}}$$

vnde occaſionem arripimus huiusmodi integrationes generalius inueſtigandi.

Exemplum 4.

1040. *Si ſint* P *et* Q *functiones quaecunque ipſius* u *et capiatur*

$$y = P\int e^{Qx} x^{n-1} dx (a-x)^{\nu-1},$$

poſito ſcilicet poſt integrationem $x = a$, *erit hic valor ipſius* y *integrale cuiuspiam aequationis differentio-differentialis:*

$$\frac{L\,ddy}{du^2} + \frac{M\,dy}{du} + Ny = 0$$

quae quaeritur.

Ad calculum contrahendum ponamus $dP = P' du$, $dP' = P'' du$, item $dQ = Q' du$ et $dQ' = Q'' du$. Hinc erit

$$\frac{dy}{du} = P'\int e^{Qx} x^{n-1} dx (a-x)^{\nu-1} + PQ'\int e^{Qx} x^{n} dx (a-x)^{\nu-1} \text{ et}$$

$$\frac{ddy}{dx^2} = P''\int e^{Qx} x^{n-1} dx (a-x)^{\nu-1} + 2P'Q'\int e^{Qx} x^{n} dx (a-x)^{\nu-1}$$
$$+ PQ''\int e^{Qx} x^{n} dx (a-x)^{\nu-1} + PQ'Q'\int e^{Qx} x^{n+1} dx (a-x)^{\nu-1}$$

vnde colligitur $\frac{L\,ddy}{du^2} + \frac{M\,dy}{du} + Ny =$

$$\int e^{Qx} x^{n-1} dx (a-x)^{\nu-1} \left\{ \begin{array}{l} LP'' + 2LP'Q'x + LPQ''x + LPQ'Q'xx \\ + MP' + MPQ'x + NP \end{array} \right\}$$

quod

quod integrale ſtatuatur $= e^{Qx} x^n (a-x)^\nu$, ita vt euaneſcat poſito $x = a$ dum ſit $\nu > 0$, vti etiam euaneſcit caſu $x = 0$ ſi modo $n > 0$. Cum igitur huius formulae differentiale ſit

$$e^{Qx} x^{n-1} dx (a-x)^{\nu-1} (Qx(a-x) + na - (n+\nu)x)$$

cuius comparatio cum forma inuenta praebet:

$$LP'' + MP' + NP = na$$

$$2LP'Q' + LPQ'' + MPQ' = aQ - (n+\nu) \text{ et}$$

$$LPQ'Q' = -Q \text{ ergo } L = \frac{-Q}{PQ'Q'} \text{ hinc}$$

$$M = \frac{aQ}{PQ'} - \frac{(n+\nu)}{PQ'} + \frac{2P'Q}{PPQ'Q'} + \frac{QQ''}{PQ'Q'Q'} \text{ et}$$

$$N = \frac{na}{P} + \frac{P''Q}{PPQ'Q'} - \frac{MP'}{P}$$

ſicque aequatio differentio - differentialis erit cognita.

Coroll. 1.

1041. Si velimus vt ſit $M = 0$ erit

$$aQ - (n+\nu) + \frac{2P'Q}{PQ'} + \frac{QQ''}{Q'Q'} = 0$$

quae per $\frac{Q'du}{Q}$ multiplicata abit in hanc

$$\frac{2dP}{P} + adQ - \frac{(n+\nu)dQ}{Q} + \frac{dQ'}{Q'} = 0$$

cuius integrale eſt

$$\frac{e^{aQ} P^2 Q'}{Q^{n+\nu}} = \text{Conſt. ſiue}$$

$$P = Ce^{-\frac{1}{2}aQ} Q^{\frac{n+\nu}{2}} \sqrt{\frac{du}{dQ}}.$$

Coroll. 2.

Coroll. 2.

1042. Sit $Q = 2\alpha u^\lambda$ erit $Q' = 2\alpha\lambda u^{\lambda-1}$ et

$$P = Ce^{-\alpha a u^\lambda} u^{\frac{\lambda(n+\nu-1)+1}{2}} \text{ hinc}$$

$$L = -\tfrac{1}{2\alpha\lambda\lambda} e^{\alpha a u^\lambda} u^{\frac{-\lambda(n+\nu+1)+1}{2}} \text{ et}$$

$$N = \tfrac{na}{P} + \tfrac{QddP}{PPaQ'^2}, \text{ at eſt}$$

$$\frac{Q}{dQ'^2} = \frac{u^{-\lambda+2}}{2\alpha\lambda\lambda du^2} \text{ et ob}$$

$$\tfrac{dP}{P} = -\alpha\lambda a u^{\lambda-1} du + \tfrac{\lambda(n+\nu-1)+1}{2} \cdot \tfrac{du}{u} \text{ erit}$$

$$\tfrac{ddP}{P} = -\alpha\lambda(\lambda-1) a u^{\lambda-2} du^2 - \tfrac{\lambda(n+\nu-1)-1}{2} \cdot \tfrac{du^2}{uu} + \alpha\alpha\lambda\lambda a a u^{2\lambda-2} du^2$$
$$-\alpha\lambda a(\lambda(n+\nu-1)+1) + \tfrac{(\lambda(n+\nu-1)+1)^2}{4}$$

ſeu

$$\tfrac{ddP}{P} = \alpha\alpha\lambda\lambda a a u^{2\lambda-2} du^2 - \alpha\lambda\lambda(n+\nu) a u^{\lambda-1} du^2 + \tfrac{\lambda\lambda(n+\nu-1)^2-1}{4} \cdot \tfrac{du^2}{uu}$$

hinc

$$na + \tfrac{Q}{dQ'^2} \cdot \tfrac{ddP}{P} = \tfrac{1}{2}\alpha a a u^\lambda + \tfrac{1}{2}(n-\nu)a + \tfrac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda} u^{-\lambda} \text{ et}$$

$$N = e^{\alpha a u^\lambda} u^{\frac{-\lambda(n+\nu-1)-1}{2}} \left(\tfrac{1}{2}\alpha a a u^\lambda + \tfrac{1}{2}(n-\nu)a + \tfrac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda} u^{-\lambda}\right).$$

Coroll. 3.

1043. Hinc erit

$$\tfrac{N}{L} = -2\alpha\lambda\lambda u^{\lambda-2}\left(\tfrac{1}{2}\alpha a a u^\lambda + \tfrac{1}{2}(n-\nu)a + \tfrac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda} u^{-\lambda}\right)$$

et huius aequationis:

$$\tfrac{ddy}{du^2} = y\left(\alpha\alpha\lambda\lambda a a u^{2\lambda-2} + \alpha\lambda\lambda(n-\nu) a u^{\lambda-2} + \tfrac{\lambda\lambda(n+\nu-1)^2-1}{4uu}\right)$$

inte-

integrale eſt

$$y = e^{-\alpha a u^\lambda}\, u^{\frac{\lambda(n+\nu-1)+1}{2}} \int e^{2\alpha u^\lambda x}\, x^{n-1}\, dx (a-x)^{\nu-1}.$$

Ponamus $\alpha = \frac{1}{\lambda}$; $\lambda(n-\nu) = f$ ſeu $\nu = n - \frac{f}{\lambda}$ et $\frac{\lambda\lambda(n+\nu-1)^2 - 1}{4} = g$

vnde fit

$$n = \frac{f + \lambda + \surd(1+4g)}{2\lambda} \text{ et } \nu = \frac{-f + \lambda + \surd(1+4g)}{2\lambda}$$

et huius aequationis

$$ddy = y\,du^2 (aau^{2\lambda-2} + afu^{\lambda-2} + gu^{-2})$$

integrale eſt [illegible] $\int e$ [illegible] x [illegible] 2λ [illegible] $dx(a-x)$ [illegible] 2λ [illegible].

Coroll. 4.

1044. Si ponamus $\alpha = \frac{-1}{\lambda}$; $\lambda(n-\nu) = -f$ et $\frac{\lambda\lambda(n+\nu-1)^2 - 1}{4} = g$ erit

$$n = \frac{-f + \lambda + \surd(1+4g)}{2\lambda} \text{ et } \nu = \frac{f + \lambda + \surd(1+4g)}{2\lambda},$$

vnde huius aequationis quae cum praecedente convenit

$$ddy = y\,du^2 (aau^{2\lambda-2} + afu^{\lambda-2} + gu^{-2})$$

integrale erit

$$y = e^{\frac{a}{\lambda} u^\lambda}\, u^{\frac{1+\surd(1+4g)}{2}} \int e^{\frac{-2x}{\lambda} u^\lambda}\, x^{\frac{-f-\lambda+\surd(1+4g)}{2\lambda}}\, dx (a-x)^{\frac{+f-\lambda+\surd(1+4g)}{2\lambda}}$$

vbi neceſſe eſt ſit $n > 0$ et $\nu > 0$.

Exemplum 5.

1045. *Si ponamus* $y = \int dx(aa-xx)^{\nu-1}\cos\alpha u^{\lambda}x$, *posito post integrationem* $x=a$, *ut* y *aequetur certae functioni ipsius* u, *inuenire aequationem differentio-differentialem, cui ea satisfaciat.*

Cum sit

$$\frac{dy}{du} = -\alpha\lambda u^{\lambda-1}\int x\,dx(aa-xx)^{\nu-1}\,\text{fin.}\,\alpha u^{\lambda}x \text{ et}$$

$$\frac{ddy}{du^2} = \int dx(aa-xx)^{\nu-1}\left\{\begin{array}{l} N\cos.\alpha u^{\lambda}x - \alpha\lambda \text{[illegible]} \\ -\alpha\alpha\lambda\lambda L u^{2\lambda-2}xx\cos.\alpha u^{\lambda}x. \end{array}\right\}$$

Fingatur integrale $=(aa-xx)^{\nu}\,\text{fin.}\,\alpha u^{\lambda}x$, quod euanescit posito tam $x=0$ quam $x=a$, reperiturque comparatione instituta

$$L = \frac{u^{-\lambda+2}}{\alpha\lambda\lambda};\quad M = \frac{2\lambda\nu-\lambda+1}{\alpha\lambda\lambda}u^{-\lambda+1};\quad N = \alpha aau^{\lambda}.$$

Quare huius aequationis

$$\frac{ddy}{du^2} + (2\lambda\nu-\lambda+1)\frac{dy}{u\,du} + \alpha\alpha\lambda\lambda aau^{2\lambda-2}y = 0$$

integrale est

$$y = \int dx(aa-xx)^{\nu-1}\cos.\alpha u^{\lambda}x.$$

Coroll. 1.

Coroll. 1.

1046. Si ergo ſit $\nu = \frac{\lambda - 1}{2\lambda}$ et $\alpha = \frac{1}{\lambda}$, huius aequationis

$$\frac{ddy}{du^2} + aau^{2\lambda - 2}y = 0$$

integrale eſt

$$y = \int dx\,(aa - xx)^{\frac{-\lambda - 1}{2\lambda}} \operatorname{cof.} \tfrac{1}{\lambda} u^\lambda x$$

ſi quidem poſt integrationem ſtatuatur $x = a$, integrali ita ſumto vt euaneſcat poſito $x = 0$.

Coroll. 2.

1047. Si igitur ſit $\frac{-\lambda - 1}{2\lambda} = i$ numero integro, ſeu $\lambda = \frac{-1}{2i + 1}$, huius aequationis

$$ddy + aau^{\frac{-4i - 4}{2i + 1}} ydu^2 = 0$$

integrale eſt

$$y = \int dx\,(aa - xx)^i \operatorname{cof.} \tfrac{1}{\lambda} u^\lambda x$$

quod reuera exhiberi poteſt. Prodeunt ſcilicet caſus integrabiles ſupra indicati.

Scholion.

1048. Cum poſuerimus $y = \int V dx$ exiſtente V functione quacunque ipſarum u et x, quarum autem in hac integratione ſola x vt variabilis tractatur, non opus eſt abſolute integrale ita determinari,

vt euanefcat pofito $x=0$, fed fufficit vt certo quodam cafu $x=b$ euanefcat, quo facto fi porro ponatur $x=a$, vt y aequetur functioni cuipiam ipfius u, quam per quadraturas affignare licet, quandoquidem hic integrationem formularum fimplicium nobis concedi iure poftulamus. Atque hic valor ipfius y per u expreffus integrale exhibebit cuiusdam aequationis differentio - differentialis

$$L\,ddy + M\,du\,dy + N\,y\,du^2 = U\,du^2,$$

vbi autem neceffe eft, vt haec formula

$$\int dx\left(L\left(\frac{ddV}{du^2}\right) + M\left(\frac{dV}{du}\right) + NV\right)$$

integrari actu poffit, quod integrale itidem ita eft capiendum, vt euanefcat pofito $x=b$, tum vero pofito $x=a$, id fiat $=U$.

Problema 131.

1049. Si fuerint P et Q functiones ipfius x, at K functio ipfius u, ac ponatur

$$y = \int P\,dx\,(K+Q)^n,$$

integrali ita fumto vt euanefcat cafu $x=b$, tum vero ftatuatur $x=a$, vt pro y prodeat functio ipfius u, inuenire aequationem differentio - differentialem inter y et u, cui ille valor ipfius y fatisfaciat.

Solutio.

Solutio.

Sit $dK = K' du$ et $dK' = K'' du$, et ob $y = \int (K+Q)^n P dx$ erit $\frac{dy}{du} = \int n K' (K+Q)^{n-1} P dx$, ac denuo differentiando

$$\frac{ddy}{du^2} = \int (n K'' (K+Q)^{n-1} + n(n-1) K' K' (K+Q)^{n-2}) P dx$$

vnde si L, M, N denotent functiones ipsius u, erit haec expressio $\frac{L\,ddy}{du^2} + \frac{M\,dy}{du} + Ny = \int P dx (K+Q)^{n-2} \times$

$$(N(K+Q)^2 + nMK'(K+Q) + nLK''(K+Q) + n(n-1)LK'K')$$

$$= \int P dx (K+Q)^{n-2} \left\{ \begin{array}{l} NKK + nMKK' + nLKK'' + n(n-1)LK'K' \\ + 2NKQ + nMK'Q + nLK''Q + NQQ \end{array} \right\}$$

quae cum debeat esse integrabilis, statuatur integrale $= R(K+Q)^{n-1} +$ Const. ita vt euanescat posito vt ante $x = b$, vbi R sit functio ipsius x tantum. Cuius formae differentiale quia est

$$(K+Q)^{n-2} (K dR + Q dR + (n-1) R dQ)$$

oportet sit:

$$(NKK + nMKK' + nLKK'' + n(n-1)LK'K') P dx$$
$$+ (2NK + nMK' + nLK'') PQ dx + NPQQ dx$$
$$= K dR + Q dR + (n-1) R dQ$$

Hic ergo duplicis generis termini adesse debent, alii ab u plane liberi, alii vero functione K affecti, quos deinceps seorsim aequari conueniet. Hunc in finem ponamus:

$NKK + nMKK' + nLKK'' + n(n-1)LK'K' = A + \alpha K$

$2NK + nMK' + nLK'' = B + \beta K$ et

$N = C + \gamma K$.

Ex binis prioribus elidendo M colligitur:

$$-NKK+n(n-1)LK'K'=A+\alpha K-BK-\beta KK$$

vnde ob $N=C+\gamma K$ concluditur

$$L=\frac{A+(\alpha-B)K-(\beta-C)KK+\gamma K^3}{n(n-1)K'K'}$$

hincque

$$M=\frac{B+\beta K-2NK-nLK''}{nK'}$$

ita vt ex functione K litterae L, M et N determinantur dum A, α, B, β, C, γ constantes quascunque denotant. Nunc autem superest vt efficiatur:

$$(A+\alpha K)Pdx+(B+\beta K)PQdx+(C+\gamma K)PQQdx$$
$$=KdR+QdR+(n-1)RdQ$$

vnde duplicis generis terminos seorsim aequando fit

$$Pdx(A+BQ+CQQ)=QdR+(n-1)RdQ$$
$$Pdx(\alpha+\beta Q+\gamma QQ)=dR$$

ideoque

$$\frac{A+BQ+CQQ}{\alpha+\beta Q+\gamma QQ}=Q+\frac{(n-1)RdQ}{dR}\text{, seu}$$
$$\frac{(n-1)RdQ}{dR}=\frac{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3}{\alpha+\beta Q+\gamma QQ}\text{ ergo}$$
$$\frac{dR}{R}=\frac{(n-1)dQ(\alpha+\beta Q+\gamma QQ)}{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3}$$

vnde ex functione Q functio R definitur: tum vero erit

$$Pdx=\frac{(n-1)RdQ}{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^3}.$$

Abeat

Abeat iam integrale illud $R(K+Q)^{n-1}+$ Conſt. in functionem U poſito $x=a$, ac valor initio aſſumtus

$$y=\int\frac{(n-1)R\,dQ(K+Q)^{n}}{A+(B-\alpha)Q+(C-\beta)QQ-\gamma Q^{3}}$$

erit integrale huius aequationis differentio - differentialis:

$$L\,ddy+M\,dudy+N\,ydu^{2}=U\,du^{2}.$$

Coroll. 1.

quatione

$$\frac{dR}{R}=\frac{(n-1)dx(\alpha+\beta x+\gamma xx)}{A+(B-\alpha)x+(C-\beta)xx-\gamma x^{3}},$$

eritque pro K functione quacunque ipſius u aſſumta:

$$y=(n-1)\int\frac{R\,dx(K+x)^{n}}{A+(B-\alpha)x+(C-\beta)xx-\gamma x^{3}},$$

in quo, integrali ita ſumto vt poſito $x=b$ euaneſcat, deinceps ſtatui debet $x=a$.

Coroll. 2.

1051. Ex functione autem K aequatio differentio - differentialis ita formatur, vt ſit

$$L=\frac{A-(B-\alpha)K+(C-\beta)KK+\gamma K^{3}}{2(n-1)dK^{2}}du^{2};\quad M=\frac{B-(2C-\beta)K-3\gamma K^{2}}{n\,dK}du$$

$$-\frac{L\,ddK}{du\,dK}\ \text{et}\ N=C+\gamma K.$$

Deinde

Deinde in expressione $R(K+x)^{n-1}+$ Const. ita constituta, vt posito $x=b$ euanescat, ponatur $x=a$, et functio ipsius u inde resultans vocetur U, eritque aequatio differentio - differentialis

$$Lddy+Mdudy+Nydu^2=Udu^2.$$

Coroll. 3.

1052. Si expressio $R(K+x)^{n-1}+$ Const. ita sit comparata, vt vtroque casu $x=b$, et $x=a$ euanescat, seu potius hi termini integrationis ita constituantur, vt hoc eueniat, formula pro y assumta satisfaciet huic aequationi …

Problema 132.

1053. Si fuerint P, Q functiones ipsius x, at K functio ipsius u, ac ponatur $y=\int e^{KQ}Pdx$, integrali ita sumto vt euanescat casu $x=b$, tum vero ponatur $x=a$, et y aequabitur functioni ipsius u, quae satisfaciet cuipiam aequationi differentio - differentiali, quam inuenire oportet.

Solutio.

Cum sit $y=\int e^{KQ}Pdx$ erit

$\frac{dy}{du}=\int e^{KQ}K'PQdx$, et $\frac{ddy}{du^2}=\int e^{KQ}Pdx(K''Q+K'K'QQ)$,

vnde

vnde fit

$$\frac{Lddy}{dx^2}+\frac{Mdy}{dx}+Ny=\int e^{KQ}Pdx(N+MK'Q+LK''Q+LK'K'QQ)$$

cuius integrale statuatur $e^{KQ}R+$ Const. quae expressio euanescat. posito $x=b$, fierique oportet

$$dR+KRdQ=Pdx(N+(MK'+LK'')Q+LK'K'QQ)$$

et ob rationes ante allegatas faciamus

$$LK'K'=A+\alpha K;\quad MK'+LK''=B+\beta K;\quad N=C+\gamma K$$

eritque

$$L=\frac{A+\alpha K}{K'K'}\ \text{et}\ M=\frac{B+\beta K}{K'}-\frac{LK''}{K'}$$

atque obtinebimus has aequationes

$$dR=Pdx(C+BQ+AQQ)$$

$$RdQ=Pdx(\gamma+\beta Q+\alpha QQ)$$

vnde colligitur

$$\frac{dR}{R}=\frac{dQ(C+BQ+AQQ)}{\gamma+\beta Q+\alpha QQ}.$$

inuentaque functione R erit

$$Pdx=\frac{RdQ}{\gamma+\beta Q+\alpha QQ}$$

ita vt sit

$$y=\int e^{KQ}\frac{RdQ}{\gamma+\beta Q+\alpha QQ}.$$

 Si

Si iam expreſſio $e^{KQ}R+\text{Conſt.}$ poſito $x=a$ abeat in functionem U, aequatio differentio-differentialis, cui hoc integrale conuenit, erit

$$Lddy+Mdudy+Nydu^2=Udu^2.$$

Coroll. 1.

1054. Hic pro Q ſcribere licet x vt ante, vnde fit

$$\frac{dR}{R}=\frac{dx(C+Bx+Axx)}{\gamma+\beta x+\alpha xx} \text{ et } y=\int e^{Kx}\cdot\frac{Rdx}{\gamma+\beta x+\alpha xx};$$

et U oritur ex forma $e^{Kx}R+\text{Conſt.}$ poſito $x=a$. Valor autem ipſius R, pro ratione coefficientium α, β, γ varias formas induere poteſt.

Coroll. 2.

1055 Pro K autem quaecunque functio ipſius u accipi poteſt, a cuius indole aequatio differentio-differentialis pendet. Erit autem $L=\frac{A+\alpha K}{dK^2}du^2$; $M=\frac{B+\beta K}{dK}du-\frac{(A+\alpha K)duddK}{dK^3}$ et $N=C+\gamma K$, vnde aequatio differentio-differentialis eſt

$$\frac{(A+\alpha K)ddy}{dK^2}+\frac{(B+\beta K)dy}{dK}-\frac{(A+\alpha K)ddKdy}{dK^3}+(C+\gamma K)y=U.$$

Coroll. 3.

1056. Cum hic etiam u ex calculo excedat, perinde eſt cuiusmodi functio eius pro K aſſumatur, quin etiam ſine detrimento amplitudinis poni poteſt $K=u$,

$K=u$, dummodo ratio elementi, quod conſtans aſſumitur, habeatur.

Scholion 1.

1057. Si ergo ſumatur $K=u$, atque elementum du ſumatur conſtans, vt fiat $ddK=0$, hinc iſta aequatio conſtrui poteſt:

$$\frac{(A+\alpha u)ddy}{dv^2}+\frac{(B+\beta u)dy}{du}+(C+\gamma u)y=U$$

exiſtente U eiusmodi functione ipſius u, quam deſcripſimus. Simili autem modo ex praecedente problemate conſtrui poteſt haec aequatio

$$(A-(B-\alpha)u+(C-\beta)uu+\gamma u^3)\frac{ddy}{du^2}+(n-1)(B-(2C-\beta)u-2\gamma uu)\frac{dy}{du}$$
$$+n(n-1)(C+\gamma u)y=U$$

quae aeque late patere eſt cenſenda, ac ſi functionem quamcunque ipſius u loco K ſcripſiſſemus. Hinc enim loco u ſcribendo functionem quamcunque ipſius t ac dt pro conſtante ſumendo omnes illae formae deriuari poſſunt. Ex quo haec aequatio multo latius patet illa, quam ſupra in genere per ſeries infinitas reſoluimus. Plerumque autem hae aequationes ita ſunt comparatae, vt earum integratio aliis methodis expediri haud poſſit, quocirca haec methodus omnino digna videtur, ad quam vlterius excolendam Geometrae omnes vires intendant.

Scho-

Scholion 2.

1058. In inueſtigatione huiusmodi conſtructionum ita ſum verſatus, vt primo quaſi per coniecturam formulam quandam differentialem $\int V dx = y$, in qua V erat certa functio ipſarum u et x, vbi autem u vt conſtans ſpectabatur, aſſumſeram, indeque dato ipſi x valore tributo pertigerim ad aequationem differentio-differentialem inter u et y, cui formula illa aſſumta ſatisfaceret. Hic autem obſeruandum eſt illam formulam integralem non prorſus ab arbitrio noſtro pendere, ſed certa quadam indole praeditam eſſe debere, vt euolutione facta res perducatur ad aequationem differentialem ſecundi gradus. Quamdiu autem hanc electionem ſoli coniecturae permittimus, perpaucae huiusmodi formulae menti ſe offerunt, quae ad ſcopum propoſitum perducant: multoque minus ſperare licet, vt hoc modo vnquam ad datam aequationem differentio-differentialem perueniamus, caſuique potiſſimum tribuendae videntur conſtructiones, quas hic tradidimus. Cum igitur longiſſime adhuc ſimus remoti a ſolutione problematis, quo propoſita quadam aequatione differentio-differentiali quaeritur formula illa eius integrationem ſuppeditans, quod problema, an vnquam ſolutionem ſit nacturum, admodum incertum videtur; eo magis opera eſt adhibenda, vt ſaltem pro caſibus particularibus inueſtigationem formulae integrantis ex indole aequatio-

nis

nis propositae deriuare conemur, sicque quodam modo viam ad solutionem directam paremus. Ad hoc autem series infinitae, per quas huiusmodi aequationes supra resoluere docuimus, vtiliter adhiberi possunt; vnde in sequenti capite methodum exponam ex serie infinita solutionem cuiuspiam aequationis differentio-differentialis continente, formulam illam integralem inuestigandi.

CAPVT XI.

DE CONSTRVCTIONE AEQVATIONVM DIFFERENTIO - DIFFERENTIALIVM EX EARVM RESOLVTIONE PER SERIES INFINITAS PETITA.

Problema 133.

1059.

Proposita ſerie infinita

$$A + Bs + Cs^2 + \ldots\ldots + Ms^{i-1} + Ns^i + \text{etc.}$$

in qua ſit $B = \frac{0m+b}{1n+k}A$; $C = \frac{1m+b}{2n+k}B$; $D = \frac{2m+b}{3n+k}C$ et in genere $N = \frac{(i-1)m+b}{in+k}M$, eius ſummam per formulam integralem exprimere.

Solutio.

Ponatur ſumma quaeſita $= z$ ita vt ſit

$$z = A + Bs + Cs^2 + Ds^3 + \ldots + Ms^{i-1} + Ns^i + \text{etc.}$$

eritque differentiando:

$$\frac{sdz}{ds} = 0A + 1Bs + 2Cs^2 + 3Ds^3 + \ldots + (i-1)Ms^{i-1} + iNs^i \text{etc.}$$

ex

ex cuius combinatione cum praecedente oritur:

$$\frac{msdz}{ds}+bz=bA+(m+b)Bs+(2m+b)Cs^2+\dots+((i-1)m+b)Ms^{i-1}$$
$$+(im+b)Ns^i \text{ etc.}$$

Deinde vero etiam simili modo est

$$\frac{nsdz}{ds}+kz=kA+(n+k)Bs+(2n+k)Cs^2+\dots+((i-1)n+k)Ms^{i-1}$$
$$+(in+k)Ns^i \text{ etc.}$$

ergo ob

$$(n+k)B=bA, \dots$$

$$\frac{nsdz}{ds}+kz=kA+bAs+(m+b)Bs^2+(2m+b)Cs^3+\text{etc.}$$

vnde manifesto conficitur:

$$\frac{nsdz}{ds}+kz=kA+\frac{mssdz}{ds}+bsz \text{ seu}$$

$$sdz(n-ms)+zds(k-bs)=kAds$$

hoc est

$$dz+\frac{zds(k-bs)}{s(n-ms)}=\frac{kAds}{s(n-ms)}.$$

Cum nunc sit

$$\frac{ds(k-bs)}{s(n-ms)}=\frac{kds}{ns}+\frac{(mk-nb)ds}{n(n-ms)},$$

aequatio ista integrabilis fit multiplicata per

$$s^{\frac{k}{n}}(n-ms)^{\frac{nb-mk}{mn}} \text{ proditque}$$

$$(n-ms)^{\frac{nb-mk}{mn}}s^{\frac{k}{n}}z=Ak\int s^{\frac{k}{n}-1}ds(n-ms)^{\frac{nb-mk}{mn}-1}$$

quod integrale ita capi oportet, vt posito $s=0$ fiat $z=A$, quo obseruato habebimus:

$$z=Aks^{-\frac{k}{n}}(n-ms)^{\frac{mk-nb}{mn}}\int s^{\frac{k}{n}-1}ds(n-ms)^{\frac{nb-mk}{mn}-1}.$$

Coroll. 1.

Coroll. 1.

1060. Peculiari ſolutione eget caſus $m = 0$, quo fit:

$$dz + \frac{z\,ds\,(k - bs)}{ns} = \frac{A\,k\,ds}{ns},$$

quae per $s^{\frac{k}{n}} e^{\frac{-bs}{n}}$ multiplicata praebet

$$e^{\frac{-bs}{n}} s^{\frac{k}{n}} z = \frac{Ak}{n} \int e^{\frac{-bs}{n}} s^{\frac{k}{n} - 1} ds$$

$$z = \frac{Ak}{n} e^{\frac{bs}{n}} s^{\frac{-k}{n}} \int e^{\frac{-bs}{n}} s^{\frac{k}{n} - 1} ds$$

integrali ita ſumto, vt fiat $z = A$ poſito $s = 0$.

Coroll. 2.

1061. Caſus etiam $n = 0$ ſeorſim reſolui debet, aequatio enim

$$dz + z\,ds\left(\frac{bs - k}{mss}\right) = -\frac{A\,k\,ds}{mss}$$

multiplicari debet per $s^{\frac{b}{m}} e^{\frac{k}{ms}}$, et inuenitur integrale:

$$e^{\frac{k}{ms}} s^{\frac{b}{m}} z = -\frac{Ak}{m} \int e^{\frac{k}{ms}} s^{\frac{b}{m} - 2} ds \text{ ideoque}$$

$$z = -\frac{Ak}{m} e^{\frac{-k}{ms}} s^{\frac{-b}{m}} \int e^{\frac{k}{ms}} s^{\frac{b}{m} - 2} ds.$$

Coroll. 3.

Coroll. 3.

1062. Si fuerit et $m=0$ et $n=0$, ob $N=\frac{b}{k}M$ feries noftra erit geometrica, aequatio vero noftra erit

$$zds(k-bs)=Akds \text{ feu } z=\frac{Ak}{k-bs}$$

vti natura rei manifefto poftulat.

Scholion.

1063. Imprimis hic cafus notari meretur quo eft $k=0$, et fumma z fine figno integrali exprimi poteft; erit namque

$$(n-ms)^{\frac{b}{m}}z=\text{Conft.}$$

et quia fi $s=0$ fieri debet $z=A$ erit Conft. $=An^{\frac{b}{m}}$ ideoque

$$z=An^{\frac{b}{m}}(n-ms)^{\frac{-b}{m}} \text{ feu } z=A\left(\frac{n}{n-ms}\right)^{\frac{b}{m}}$$

vel etiam

$$z=A\left(1-\frac{ms}{n}\right)^{\frac{-b}{m}}.$$

At vero integratio etiam fuccedit cafu quo $k=n$ erit enim

$$(n-ms)^{\frac{b}{m}-1}sz=An\int ds(n-ms)^{\frac{b}{m}-2}$$

quod integrale eft Conft. $-\frac{An(n-ms)^{\frac{b}{m}-1}}{b-m}$

et

et quia posito $s=0$ fit $z=A$, erit $0=\text{Const.}-\frac{A}{b-m}.n^{\frac{b}{m}}$, hincque

$$z=\frac{An}{(b-m)s}\left(\left(\frac{n}{n-ms}\right)^{\frac{b}{m}-1}-1\right)=\frac{An}{(b-m)s}\left(\left(1-\frac{ms}{n}\right)^{1-\frac{b}{m}}-1\right)$$

Porro perspicuum est, integrationem expediri posse casu $k=2n$ quo cum sit

$$(n-ms)^{\frac{b}{m}-2}ssz=2An\int sds(n-ms)^{\frac{b}{m}-3}$$

erit hoc integrale

$$=\text{Const.}-\frac{2Ans}{b-2m}(n-ms)^{\frac{b}{m}-2}+\frac{2An}{b-2m}\int ds(n-ms)^{\frac{b}{m}-2}.$$

$$=\text{Const.}-\frac{2Ans}{b-2m}(n-ms)^{\frac{b}{m}-2}-\frac{2An(n-ms)^{\frac{b}{m}-1}}{(b-m)(b-2m)}$$

vbi $\text{Const.}=\frac{2An^{\frac{b}{m}}}{(b-m)(b-2m)}$, ideoque

$$z=\frac{2Ann}{(b-m)(b-2m)ss}\left(\left(\frac{n}{n-ms}\right)^{\frac{b}{m}-2}-1-\frac{(b-2m)s}{n}\right).$$

similique modo etiam integratio casibus $k=3n$, $k=4n$ etc. absoluetur.

Problema 134.

1064. Proposita huiusmodi serie infinita:

$$A\mathfrak{A}+B\mathfrak{B}u+C\mathfrak{C}u^2+D\mathfrak{D}u^3+\ldots+M\mathfrak{M}u^{i-1}+\mathfrak{N}Nu^i+\text{etc.}$$

coeffi-

coefficientium lege exiſtente

$$B=\frac{0m+b}{1n+k}A;\ C=\frac{1m+b}{2n+k}B;\ D=\frac{2m+b}{3n+k}C\ldots\ldots N=\frac{(i-1)m+b}{in+k}M$$

$$\mathfrak{B}=\frac{0\mu+\eta}{1\nu+\theta}\mathfrak{A};\ \mathfrak{C}=\frac{1\mu+\eta}{2\nu+\theta}\mathfrak{B};\ \mathfrak{D}=\frac{2\mu+\eta}{3\nu+\theta}\mathfrak{C}\ldots\ldots \mathfrak{N}=\frac{(i-1)\mu+\eta}{i\nu+\theta}\mathfrak{M}$$

eius ſummam per formulam integralem exprimere.

Solutio.

Poſita ſumma

$$y=A\mathfrak{A}+B\mathfrak{B}u+C\mathfrak{C}u^2+D\mathfrak{D}u^3+E\mathfrak{E}u^4+\text{etc.}$$

conſideretur ſeries hoc modo formata:

$$z=A+Bux+Cu^2x^2+Du^3x^3+Eu^4x^4+\text{etc.}$$

cuius ſumma poſito $ux=s$ eſt vt modo inuenimus:

$$z=Aks^{-\frac{k}{n}}(n-ms)^{\frac{mk-nb}{mn}}\int s^{\frac{k}{n}-1}ds(n-ms)^{\frac{nb-mk}{mn}-1}$$

integrali ita determinato, vt poſito $s=0$ fiat $z=A$.

Formetur huiusmodi formula integralis:

$$V=\int Pz\,dx=\int P\,dx(A+Bux+Cu^2x^2+Du^3x^3+\text{etc.})$$

in qua u ſpectetur vt conſtans, pro P autem eiusmodi functio ipſius x accipiatur, vt fiat

$$\int Px\,dx=\frac{\mathfrak{B}}{\mathfrak{A}}\int P\,dx;\ \int Px^2dx=\frac{\mathfrak{C}}{\mathfrak{B}}\int Px\,dx;\ \int Px^3dx=\frac{\mathfrak{D}}{\mathfrak{C}}\int Px^2dx\ \text{etc.}$$

poſtquam ſcilicet in his integralibus data lege ſumtis variabili x datus quidem valor fuerit tributus. Cum igitur hinc ſit

$$\int Px\,dx=\frac{\mathfrak{B}}{\mathfrak{A}}\int P\,dx;\ \int Px^2dx=\frac{\mathfrak{C}}{\mathfrak{A}}\int P\,dx;\ \int Px^3dx=\frac{\mathfrak{D}}{\mathfrak{A}}\int P\,dx\ \text{etc.}$$

 erit

erit

$$V=(A+\frac{B\mathfrak{B}}{\mathfrak{A}}u+\frac{C\mathfrak{C}}{\mathfrak{A}}u^2+\frac{D\mathfrak{D}}{\mathfrak{A}}u^3+\text{etc.})\int P\,dx$$

vnde patet fore

$$y=\frac{\mathfrak{A}V}{\int P\,dx}=\frac{\mathfrak{A}\int Pz\,dx}{\int P\,dx}.$$

Quare cum valor ipſius z ſit cognitus, tantum ſupereſt, vt functio P ipſius x conditionibus memoratis praedita inueſtigetur. In genere autem eſſe oportet:

$$\int Px^i\,dx=\frac{\mathfrak{N}}{\mathfrak{M}}\int Px^{i-1}\,dx=\frac{(i-1)\mu+\eta}{i\nu+\theta}\int P\,dx^{i-1}\,dx$$

quae aequalitas, cum ſufficiat vt tantum certo quodam caſu quo ipſi x datus tribuitur valor, ſubſiſtat, ponamus in genere eſſe

$$(i\nu+\theta)\int Px^i\,dx=((i-1)\mu+\eta)\int Px^{i-1}\,dx+x^iQ$$

ita vt pro terminis integralibus ſit $Q=0$. Differentiando ergo habebimus

$$(i\nu+\theta)Px^i\,dx=(i\mu-\mu+\eta)Px^{i-1}\,dx+x^i\,dQ+ix^{i-1}Q\,dx$$

ſeu per x^{i-1} diuidendo:

$$(i\nu+\theta)Px\,dx=(i\mu-\mu+\eta)P\,dx+x\,dQ+iQ\,dx$$

quae aequalitas, cum pro omnibus valoribus ipſius i aeque ſubſiſtere debeat, hinc duas adipiſcimur aequationes:

$$\nu Px\,dx=\mu P\,dx+Q\,dx \quad\text{et}\quad \theta Px\,dx=(\eta-\mu)P\,dx+x\,dQ$$

vnde

vnde duplici modo colligimus:

$$Pdx = \frac{Q\,dx}{\nu x-\mu} \text{ et } Pdx = \frac{x\,dQ}{\theta x-(\eta-\mu)}$$

sicque alterum valorem per alterum diuidendo

$$\frac{x\,dQ}{Q\,dx} = \frac{\theta x+\mu-\eta}{\nu x-\mu} \text{ seu } \frac{dQ}{Q} = \frac{dx(\theta x+\mu-\eta)}{x(\nu x-\mu)}$$

quae euoluitur in

$$\frac{dQ}{Q} = \frac{\eta-\mu}{\mu x}dx + \frac{\mu\theta+\mu\nu-\eta\nu}{\mu(\nu x-\mu)}dx$$

hinc integrando elicitur:

$$Q = x^{\frac{\eta}{\mu}-1}(\nu x-\mu)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1} \text{ in Const.}$$

seu $Q = -x^{\frac{\eta}{\mu}-1}(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1}$

vnde fit

$$Pdx = x^{\frac{\eta}{\mu}-1}dx(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}.$$

Cum igitur sit

$$(i\nu+\theta)\int Px^i dx = ((i-1)\mu+\eta)\int Px^{i-1}dx - x^{i+\frac{\eta}{\mu}-1}(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1}$$

si haec integralia ita capiantur vt euanescant posito $x=0$, tum vero statuatur $x=\frac{\mu}{\nu}$, fiet vti hypothesis nostra postulat

$$\int Px^i dx = \frac{(i-1)\mu+\eta}{i\nu+\theta}\int Px^{i-1}dx$$

at vero in hunc finem necesse est, vt sit

$$i+\frac{\eta}{\mu}-1>0 \text{ et } \frac{\theta\mu-\eta\nu}{\mu\nu}+1>0.$$

Quae conditio ſi locum habeat, ſeriei propoſitae ſumma ita exprimetur, vt ſit:

$$y\int x^{\frac{\eta}{\mu}-1}dx(\mu-\nu x)^{\frac{\theta\mu+\eta\nu}{\mu\nu}}=\mathfrak{A}\int x^{\frac{\eta}{\mu}-1}zdx(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$$

exiſtente

$$z=Aks^{\frac{-k}{n}}(n-ms)^{\frac{mk-nb}{mn}}\int s^{\frac{k}{n}-1}ds(n-ms)^{\frac{nb-mk}{mn}-1}$$

integrali hoc ita ſumto, vt fiat $z=A$ poſito $s=0$. Hoc autem integrali inuento pro s ſcribatur ux, et hoc valore ipſius z in illa formula ſubſtituto quantitatem u tanquam conſtantem tractari oportet, quoad illae integrationes lege praeſcripta fuerint abſolutae, tum enim pro y prodibit functio ipſius u ſummam ſeriei propoſitae exprimens.

Coroll. 1.

1065. Quia in geminatis coefficientibus noſtrae ſeriei ſimilis lex progreſſionis aſſumitur, ſingulas ſeries A, B, C, D etc. et $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. inter ſe permutare licet, vnde hoc methodo duplex formula ſummam ſeriei exprimens obtinetur.

Coroll. 2.

1066. Etſi functio Q non in calculum ingreditur, eam tamen noſſe oportet, quoniam ex eius indole termini integrationis conſtitui debent,

ita

ita vt pro vtroque fiat $Q=0$. Hi scilicet termini sunt $x=0$ et $x=\frac{\mu}{\nu}$ dum fuerit

$$i+\frac{\eta}{\mu}-1>0 \text{ et } \frac{\theta\mu-\eta\nu}{\mu\nu}+1>0,$$

vbi i est numerus integer positiuus.

Coroll. 3.

1067. Pro functione Q casus quo vel $\mu=0$ vel $\nu=0$ seorsim sunt euoluendi. Priore quo $\mu=0$ est

$$\frac{dQ}{Q}=\frac{dx(\theta x-\eta)}{\nu xx}=\frac{\theta dx}{\nu x}-\frac{\eta dx}{\nu xx}, \text{ vnde fit}$$

$$Q=e^{\frac{\eta}{\nu x}}x^{\frac{\theta}{\nu}}.$$

Posteriori quo $\nu=0$ est

$$\frac{dQ}{Q}=\frac{dx(\theta x+\mu-\eta)}{-\mu x}=-\frac{\theta dx}{\mu}+\frac{\eta-\mu}{\mu}.\frac{dx}{x}$$

ideoque

$$Q=e^{\frac{-\theta x}{\mu}}x^{\frac{\eta}{\mu}-1}.$$

Scholion.

1068. Constructiones hoc modo adornandae prorsus similes sunt iis, quas capite praecedente tradidimus, cum res etiam ad formulam integralem huiusmodi $\int V dx$ reducatur, in qua V est functio binarum variabilium u et x, quarum illa autem u in ipsa integratione constans reputatur, post integrationem vero ipsi x datus quidam valor assignatur. Verum tamen haec constructio ad casus in superiori methodo non contentos extenditur, quandoquidem fieri

fieri potest, vt quantitas z functiones maxime transcendentes inuoluat. Viciſſim autem vidimus, methodum praecedentem ad eiusmodi aequationes applicari posse, quae per series, quales hic tractamus, euolui nequeant, vnde in Analysin ex hoc fonte haud contemnenda incrementa hauriri posse videntur.

Problema 135.

1069. Proposita aequatione differentio-differentiali

$$xx(a+bx^n)ddy+(c+ex^n)dxdy+(f+gx^n)ydx^2=0$$

valorem ipsius y per formulam integralem construere.

Solutio I.

Hanc aequationem supra (968.) ita in seriem euoluimus vt posito

$$y=x^\lambda(A+Bx^n+Cx^{2n}+Dx^{3n}+Ex^{4n}\text{etc.})$$

primo exponenti λ tribui debeat radix huius aequationis $\lambda(\lambda-1)a+\lambda c+f=0$, tum vero posito breuitatis gratia $\lambda(\lambda-1)b+\lambda e+g=b$ fit

$$B=\frac{-b}{n(na+(2\lambda-1)a+c)}A$$

$$C=\frac{-nnb-(2\lambda-1)nb-ne-b}{2n(2na+(2\lambda-1)a+c)}B$$

$$D=\frac{-4nnb-2(2\lambda-1)nb-2ne-b}{3n(3na+(2\lambda-1)a+c)}C$$

ideoque

ideoque illius feriei pofitis binis terminis contiguis indefinite $Mx^{(i-1)n}+Nx^{in}$, generaliter

$$N=\frac{-(i-1)^2nnb-(2\lambda-1)(i-1)nb-(i-1)ne-b}{in(ina+(2\lambda-1)a+c)}M$$

vbi cum denominator iam habeat factores, quales ante affumfimus, numeratorem quoque in factores refoluamus, quo nihilo aequali pofito inuenitur

$$(i-1)n=-\tfrac{1}{2}(2\lambda-1)-\tfrac{e}{2b}\pm\sqrt{\left(\tfrac{1}{4}(2\lambda-1)^2+\tfrac{(2\lambda-1)e}{2b}+\tfrac{ee}{4bb}-\tfrac{b}{b}\right)}$$

feu $(i-1)n=-\tfrac{1}{2}(2\lambda-1)-\tfrac{e}{2b}\pm\tfrac{1}{2b}\sqrt{((b-e)^2-4bg)}$.

Ponatur breuitatis gratia

$$\sqrt{((b-e)^2-4bg)}=q,$$

vt fit

$$(i-1)n=\frac{-(2\lambda-1)b-e\pm q}{2b}$$

et noftra relatio fiet

$$N=\frac{-((i-1)nb+\tfrac{1}{2}(2\lambda-1)b+\tfrac{1}{2}e-\tfrac{1}{2}q)((i-1)nb+\tfrac{1}{2}(2\lambda-1)b+\tfrac{1}{2}e+\tfrac{1}{2}q)}{inb(ina+(2\lambda-1)a+c)}M.$$

Ponamus iam $x^n=u$, et feriem inuentam ita repraefentemus

$$\frac{y}{x^\lambda}=A\mathfrak{A}+B\mathfrak{B}u+C\mathfrak{C}u^2+\ldots\ldots+M\mathfrak{M}u^{i-1}+N\mathfrak{N}u^i\text{etc.}$$

horumque duplicium coefficientium lex ita fe habebit:

$$N=\frac{(i-1)nb+\tfrac{1}{2}(2\lambda-1)b+\tfrac{1}{2}e-\tfrac{1}{2}q}{-inb}M \text{ et}$$

$$\mathfrak{N}=\frac{(i-1)nb+\tfrac{1}{2}(2\lambda-1)b+\tfrac{1}{2}e+\tfrac{1}{2}q}{ina+(2\lambda-1)a+c}\mathfrak{M}.$$

Cum

Cum igitur haec feries fimilis fit ei, quam ante conftruximus comparationem inftituamus, et habebimus:

$$m = nb\,;\ b = \tfrac{1}{2}(2\lambda - 1)b + \tfrac{1}{2}e - \tfrac{1}{2}q$$
$$n = -nb\,;\ \text{et}\ k = 0$$
$$\mu = nb\,;\ \eta = \tfrac{1}{2}(2\lambda - 1)b + \tfrac{1}{2}e + \tfrac{1}{2}q$$
$$\nu = na\ \text{ et }\ \theta = (2\lambda - 1)a + c.$$

Primum ergo quaeramus quantitatem z, et ne littera x ambiguitatem creet, loco litterae x in praecedente problemate vfurpatae vtamur littera t, fitque $ut = s$, et quia eft $k = 0$, erit per §. 1064.

$$z = A(1+s)^{\frac{-(2\lambda-1)b-e+q}{2nb}} = A(1+ut)^{\frac{-(2\lambda-1)b-e+q}{2nb}}.$$

Hoc valore inuento tractetur in fequentibus integrationibus quantitas u vt conftans, et cum quod fupra erat y hic fit $\frac{y}{x^\lambda}$, et t quod fupra erat x, erit

$$\frac{y}{x^\lambda}\int t^{\frac{\eta}{\mu}-1}dt(\mu-\nu t)^{\frac{\theta\mu+\eta\nu}{\mu\nu}} = \mathfrak{A}\int t^{\frac{\eta}{\mu}-1}z\,dt(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$$

vbi cum fit $u = x^n$ hic valor ftatim pro u fcribi poteft, vt fit

$$z = A(1+x^n t)^{\frac{-(2\lambda-1)b-e+q}{2nb}},$$

et in his integrationibus littera x vt conftans fpectari debet. Quodfi autem fuerit:

$$i + \tfrac{\eta}{\mu} - 1 > 0 \ \text{ et } \ \tfrac{\theta\mu-\eta\nu}{\mu\nu} + 1 > 0$$

haec

haec integralia ita capi debent, vt euanefcant pofito $t=0$, quo facto ipfi t tribui debet valor $t=\frac{\mu}{\nu}=\frac{b}{a}$. Cum vnitas fit minimus valor ipfius i, fufficit vt fit

$$\frac{(2\lambda-1)b+e+q}{2nb}>0\text{, tum vero }\frac{(2\lambda+1)ab+2bc-ae-aq}{2nab}+1>0.$$

Nunc vero cum

$$\int t^{\frac{\eta}{\mu}-1}dt(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$$

fiat quantitas conftans, pro y autem eius multiplum quoduis noftrae aequationi aequae fatisfaciat, eius integrale ita exprimetur:

$$y=Cx^\lambda\int t^{\frac{\eta}{\mu}-1}z\,dt(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$$

exiftente

$$z=(1+x^n t)^{\frac{-(2\lambda-1)b-e+q}{2nb}}.$$

Solutio II.

1071. Si coefficientes geminatos inter fe permutemus, vt fit

$$m=nb;\ h=\tfrac{1}{2}(2\lambda-1)b+\tfrac{1}{2}e+\tfrac{1}{2}q$$
$$n=na;\ k=(2\lambda-1)a+c$$
$$\mu=nb;\ \eta=\tfrac{1}{2}(2\lambda-1)b+\tfrac{1}{2}e-\tfrac{1}{2}q$$
$$\nu=-nb;\ \theta=0$$

fumaturque λ ex aequatione

$$\lambda(\lambda-1)a+\lambda c+f=0.$$

 Primo

Primo ponatur $x^n t = s$, et quaeratur z vt fit:

$$z = Aks^{\frac{-k}{n}}(n-ms)^{\frac{mk-nb}{mn}} \int s^{\frac{k}{n}-1} ds\,(n-ms)^{\frac{nb-mk}{mn}-1}$$

integrali ita definito, vt posito $s=0$ fiat $z=A$, qui quidem valor A est arbitrarius, tum spectata x vt constante erit

$$y = Cx^{\lambda} \int t^{\frac{\eta}{\mu}-1} z\,dt(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$$

integrali ita sumto vt euanescat posito $t=0$; tum vero facto $t=\frac{\mu}{\nu}$, si modo fuerit $\frac{\eta}{\mu}>0$ et $1-\frac{\eta}{\mu}>0$ ob $\theta=0$; vbi notetur z per hanc aequationem differentialem definiri

$$\frac{dz}{ds} = \frac{Ak - z(k-bs)}{s(n-ms)}.$$

Solutio III.

1072. Per seriem descendentem aequationem propositam ita resoluimus vt posito

$$y = x^{\lambda}(A + Bx^{-n} + Cx^{-2n} + Dx^{-3n} + \text{etc.})$$

exponens λ definiri debeat ex hac aequatione

$$\lambda(\lambda-1)b + \lambda e + g = 0,$$

tum vero posito $\lambda(\lambda-1)a + \lambda c + f = b$, fit

$$B = \frac{-b}{n(nb-(2\lambda-1)b-e)}A$$

$$C = \frac{-nna+(2\lambda-1)na+nc-b}{2n(2nb-(2\lambda-1)b-e)}B$$

et generatim

$$N = \frac{-(i-1)^2 nna + (2\lambda-1)(i-1)na + (i-1)nz + (i-1)nc - b}{in(inb-(2\lambda-1)b-e)}M$$

quae

quae aequalitas posito $\sqrt{((a-c)^2-4af)}=p$ ita per factores exhibetur

$$N=\frac{-((i-1)na-\tfrac{1}{2}(2\lambda-1)a-\tfrac{1}{2}c-\tfrac{1}{2}p)((i-1)na-\tfrac{1}{2}(2\lambda-1)a-\tfrac{1}{2}c+\tfrac{1}{2}p)}{ina(inb-(2\lambda-1)b-e)}M.$$

Quodsi iam ponamus $x^{-n}=u$, et talem seriem constituamus:

$$\frac{y}{x^\lambda}=A\mathfrak{A}+B\mathfrak{B}u+C\mathfrak{C}u^2+\ldots\ldots+M\mathfrak{M}u^{i-1}+N\mathfrak{N}u^i+\text{etc.}$$

erit $N=\dfrac{(i-1)na-\tfrac{1}{2}(2\lambda-1)a-\tfrac{1}{2}c-\tfrac{1}{2}p}{-ina}M$

et $\mathfrak{N}=\dfrac{(i-1)na-\tfrac{1}{2}(2\lambda-1)a-\tfrac{1}{2}c+\tfrac{1}{2}p}{inb-(2\lambda-1)b-e}\mathfrak{M}$

habebimus comparatione instituta, cum constructione generali:

$m=na$; $b=-\tfrac{1}{2}(2\lambda-1)b-\tfrac{1}{2}c-\tfrac{1}{2}p$

$n=-na$; et $k=0$

$\mu=na$; $\eta=-\tfrac{1}{2}(2\lambda-1)a-\tfrac{1}{2}c+\tfrac{1}{2}p$

$\nu=nb$; $\theta=-(2\lambda-1)b-e.$

Hinc posito $s=ut=x^{-n}t$ erit

$$z=A(1+s)^{\frac{-b}{m}}=A(1+x^{-n}t)^{\frac{-b}{m}}$$

quo valore inuento, si iam sola quantitas t pro variabili habeatur, orietur haec constructio:

$$y=Cx^\lambda\int t^{\frac{\eta}{\mu}-1}z\,dt(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$$

 vbi

vbi termini integrationis ita sunt constituendi vt vtroque fiat

$$t^{\frac{\eta}{\mu}}(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1}=0.$$

Solutio IV.

1073. Hic etiam coefficientes geminati permutari possunt, vt sit:

$$m=na\,;\quad b=-\tfrac{1}{2}(2\lambda-1)a-\tfrac{1}{2}c+\tfrac{1}{2}p$$
$$n=nb\,;\quad k=-(2\lambda-1)b-e$$
$$\mu=na\,;\quad \eta=-\tfrac{1}{2}(2\lambda-1)a-\tfrac{1}{2}c-\tfrac{1}{2}p$$
$$\nu=-na\,;\quad \theta=0$$

sumtoque vt ante λ ex aequatione

$$\lambda(\lambda-1)b+\lambda e+g=0$$

et posito $\sqrt{((a-c)^2-4af)}=p$, fit $s=x^{-n}t$, et quaeratur z ex hac aequatione:

$$\frac{dz}{ds}=\frac{Ak-z(k-be)}{s(n-ms)}$$

ita vt posito $s=0$ fiat $z=A$, vnde fit:

$$z=Aks^{\frac{-k}{n}}(n-ms)^{\frac{mk-nb}{mn}}\int s^{\frac{k}{n}-1}ds(n-ms)^{\frac{nb-mk}{mn}-1}$$

tum vero spectata x vt constante erit

$$y=Cx^{\lambda}\int t^{\frac{\eta}{\mu}-1}z\,dt(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$$

vbi bini termini integrationis ita sumi debent, vt vtroque fiat

$$t^{\frac{\eta}{\mu}}(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1}=0.$$

Scholion.

Scholion.

1074. Singulae hae constructiones plerumque pluribus modis exhiberi possunt, cum non solum λ duplicem valorem habere queat, sed etiam formulae radicales p et q signo ambiguo gaudeant. At hae constructiones alio modo negotium conficiunt atque praecedentes, quod quo clarius appareat, consideremus aequationem

$$xx(1-xx)ddy - x(1+xx)dxdy + xxydx^2 = 0$$

ita vt sit $a=1$, $b=-1$, $c=-1$, $e=-1$, $f=0$ et $g=1$, atque $n=2$, vnde pro duabus prioribus constructionibus habetur $\lambda(\lambda-1)-\lambda=0$, ergo vel $\lambda=0$ vel $\lambda=2$, tum vero $q=\pm 2$. Constructio ergo prima dat: $\lambda=0$, $m=-2$, $b=\mp 1$, $n=2$, $k=0$, $\mu=-2$, $\eta=\pm 1$, $\nu=2$, $\theta=-2$, vnde colligitur

$$z=(1+xxt)^{\mp\frac{1}{2}} \text{ et } y=C\int t^{\mp\frac{1}{2}-1} z\,dt(1+t)^{\pm\frac{1}{2}-1}.$$

Valeant signa inferiora, vt sit

$$z=\sqrt{(1+xxt)} \text{ et } y=C\int\frac{z\,dt}{(1+t)\sqrt{t(1+t)}}.$$

Quae formulae quomodo satisfaciant, videamus. Sumta nempe sola x variabili, fit

$$\frac{dz}{dx}=\frac{xt}{\sqrt{(1+xxt)}} \text{ et } \frac{ddz}{dx^2}=\frac{t}{(1+xxt)^{\frac{3}{2}}}$$

hinc

$$\frac{dy}{dx}=C\int\frac{xtdt}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{1}{2}}} \text{ et } \frac{ddy}{dx^2}=C\int\frac{tdt}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{3}{2}}}$$

vnde

vnde conficitur

$$xx(1-xx)\frac{ddy}{dx^2}-x(1+xx)\frac{dy}{dx}+xxy=C\int\frac{xxdt(1-xxtt)}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{3}{2}}},$$

cuius formulae integrale eſt $\frac{2Cxx\sqrt{t}}{\sqrt{(1+t)(1+xxt)}}$; quod cum euaneſcat tam caſu $t=0$ quam caſu $t=\infty$, conſtructio noſtrae aequationis

$$y=C\int\frac{zdt}{(1+t)\sqrt{t(1+t)}}=C\int\frac{dt\sqrt{(1+xxt)}}{(1+t)\sqrt{t(1+t)}}$$

ita confici debet; ſumto x pro conſtante integratio ita inſtituatur, vt integrale euaneſcat poſito $t=0$, quo facto ſtatuatur $t=\infty$, et functio ipſius x, quae pro y prodibit, ſatisfaciet aequationi propoſitae.

Sin autem ſecundam conſtructionem eligamus, ſumta $\lambda=0$, erit $m=-2$, $b=\pm 1$, $n=2$, $k=-2$, $\mu=-2$, $\eta=\mp 1$, $\nu=2$, $\theta=0$, atque z ita definiri debet ex hac aequatione

$$\frac{dz}{ds}=\frac{-2A+z(2\pm s)}{s(2+2s)},$$

exiſtente $s=xxt$, ita vt poſito $s=0$ fiat $z=A$. Valeat ſignum ſuperius, et cum ſit

$$dz-\frac{zds(2+s)}{2s(1+s)}=\frac{-Ads}{s(1+s)}$$

multiplicetur per $\frac{\sqrt{(1+s)}}{s}$ eritque integrale

$$\frac{z\sqrt{(1+s)}}{s}=\text{Conſt.}-A\int\frac{ds}{ss\sqrt{(1+s)}} \text{ ſeu}$$

$$z=A-\frac{As}{\sqrt{(1+s)}}\,l\,\frac{1+\sqrt{(1+s)}}{\sqrt{s}}+\frac{Bs}{\sqrt{(1+s)}},$$

quae

quae posito $t = 0$ dat $z = A$ quicquid sit B. Deinde est

$$y = C\int t^{\frac{1}{2}-1} z\,dt(1+t)^{-\frac{1}{2}} \quad \text{seu} \quad y = C\int \frac{z\,dt}{\sqrt{t(1+t)}},$$

qui valor quomodo satisfaciat haud facile ostendi potest; hocque magis ista methodus excoli meretur

Exemplum.

1075. *Constructiones aequationis differentio-differentialis*:

$$xx(1-xx)ddy - x(1+xx)dxdy + xxydx^2 = 0$$

ex praecedente problemate oriundas exhibere.

Ob $n = 2$, $a = 1$, $b = -1$, $c = -1$, $e = -1$ $f = 0$, et $g = 1$, pro prima constructione habemus vel $\lambda = 0$ vel $\lambda = 2$, vnde habemus.

1.) Si $\lambda = 0$, vt modo inuenimus:

$$m = -2;\ b = \mp 1;\ n = 2;\ k = 0;\ \mu = -2;\ \eta = \pm 1;$$
$$\nu = 2;\ \theta = -2$$

vnde fit

$$z = (1+xxt)^{\mp\frac{1}{2}} \quad \text{et} \quad y = C\int t^{\mp\frac{1}{2}-1} z\,dt(1+t)^{\pm\frac{1}{2}-1}$$

sicque duplex oritur constructio

altera $z = \sqrt{(1+xxt)}$ et $y = C\int \frac{z\,dt}{(1+t)\sqrt{t(1+t)}}$

altera $z = \frac{1}{\sqrt{(1+xxt)}}$ et $y = C\int \frac{z\,dt}{t\sqrt{t(1+t)}}$.

2.) Si $\lambda = 2$, erit ob $q = \pm 2$

$$m = -2;\ b = -2 \mp 1;\ n = 2;\ k = 0;\ \mu = -2;$$
$$\eta = -2 \pm 1;\ \nu = 2;\ \theta = 2$$

vnde fit

$$z = (1 + xxt)^{\frac{-2 \mp 1}{2}} \text{ et } y = Cx^2 \int t^{+1 \mp \frac{1}{2} - 1} z\,dt(1+t)^{\pm \frac{1}{2}}$$

ſicque duplex habetur conſtructio

$$\text{altera } z = (1 + xxt)^{-\frac{3}{2}} \text{ et } y = Cx^2 \int \frac{z\,dt\sqrt{(1+t)}}{\sqrt{t}}$$

$$\text{altera } z = (1 + xxt)^{-\frac{1}{2}} \text{ et } y = Cx^2 \int \frac{z\,dt\sqrt{t}}{\sqrt{(1+t)}}.$$

Pro ſecunda conſtructione generali habemus:

3.) Si $\lambda = 0$ permutando illos indices

$$m = -2;\ b = \pm 1;\ n = 2;\ k = -2;\ \mu = -2;$$
$$\eta = \mp 1;\ \nu = 2;\ \theta = 0$$

vnde poſito $xxt = s$, primo quaeratur z ex hac aequatione $\frac{dz}{ds} = \frac{-2A + z(2 \pm s)}{2s(1+s)}$ vt poſito $s = 0$ fiat $z = A$, tum vero erit

$$y = C\int t^{\pm \frac{1}{2} - 1} z\,dt(1+t)^{\mp \frac{1}{2}}.$$

Hinc ergo naſcitur duplex conſtructio:

$$\text{altera } \frac{dz}{ds} = \frac{-2A + z(2+s)}{2s(1+s)} \text{ et } y = C\int \frac{z\,dt}{\sqrt{t(1+t)}}$$

$$\text{altera } \frac{dz}{ds} = \frac{-2A + z(2-s)}{2s(1+s)} \text{ et } y = C\int \frac{z\,dt\sqrt{(1+t)}}{t\sqrt{t}}.$$

4.) Si $\lambda = 2$ habebimus:

$$m = -2;\ b = -2 \pm 1;\ n = 2;\ k = 2;\ \mu = -2;$$
$$\eta = -2 \mp 1;\ \nu = 2;\ \theta = 0$$

poſito-

positoque $xxt = s$, vt ante, quaeratur z ex aequatione:

$$\frac{dz}{ds} = \frac{2A - z(2 + (2 \mp 1)s)}{2s(1+s)} \text{ eritque}$$

$$y = Cx^2 \int t^{\pm\frac{1}{2}} z\,dt(1+t)^{\frac{-1\mp 1}{2}}$$

ideoque etiam duplex constructio elicitur:

altera $\frac{dz}{ds} = \frac{2A - z(2+s)}{2s(1+s)}$ et $y = Cx^2 \int \frac{z\,dt\sqrt{t}}{(1+t)\sqrt{(1+t)}}$

altera $\frac{dz}{ds} = \frac{2A - z(2+3s)}{2s(1+s)}$ et $y = Cx^2 \int \frac{z\,dt}{\sqrt{t(1+t)}}$

Ex solutione tertia colligimus primo

$$-\lambda(\lambda - 1) - \lambda + 1 = 0 \text{ seu } \lambda\lambda = 1$$

ideoque

$$\lambda = \pm 1 \text{ et } p = \sqrt{4} = \pm 2 \text{ ; quare}$$

5.) si capiatur $\lambda = +1$ erit

$m = 2$; $b = \mp 1$; $n = -2$; $k = 0$; $\mu = 2$; $\eta = \pm 1$;
$\nu = -2$; $\theta = 2$

hincque

$$z = (1 + \tfrac{t}{xx})^{\pm\frac{1}{2}} \text{ et } y = Cx \int t^{\pm\frac{1}{2}-1} z\,dt(1+t)^{-1\mp\frac{1}{2}}$$

ita vt iterum duplex habeatur constructio:

altera $z = \frac{1}{x}\sqrt{(xx+t)}$ et $y = Cx \int \frac{z\,dt}{(1+t)\sqrt{t(1+t)}}$

altera $z = \frac{x}{\sqrt{(xx+t)}}$ et $y = Cx \int \frac{z\,dt}{t\sqrt{t(1+t)}}$.

6.) Si capiatur $\lambda = -1$ erit

$m = 2$; $b = 2 \mp 1$; $n = -2$; $k = 0$; $\mu = 2$; $\eta = 2 \pm 1$;
$\nu = -2$; $\theta = -2$

hincque

$$z=(1+\tfrac{t}{xx})^{-1\pm\frac{1}{2}} \text{ et } y=\tfrac{C}{x}\int t^{\pm\frac{1}{2}}z\,dt(1+t)^{\mp\frac{1}{2}}$$

vnde binae constructiones fluunt:

altera $z=\dfrac{x}{\sqrt{(xx+t)}}$ et $y=\dfrac{C}{x}\int\dfrac{z\,dt\sqrt{t}}{\sqrt{(1+t)}}$

altera $z=\dfrac{x^3}{(xx+t)^{\frac{3}{2}}}$ et $y=\dfrac{C}{x}\int\dfrac{z\,dt\sqrt{(1+t)}}{\sqrt{t}}$.

Ex solutione quarta denique concludimus:

7.) Si $\lambda=\mp 1$

$m=2$; $b=\pm 1$; $n=-2$; $k=2$; $\mu=2$; $\eta=\mp 1$; $\nu=-2$; $\theta=0$.

Posito nunc $s=\frac{t}{xx}$, quaeratur z ex hac aequatione,

$$\frac{dz}{ds}=\frac{-2A+z(2\mp s)}{2s(1+s)}$$

vt posito $s=0$ fiat $z=A$, tumque erit

$$y=Cx\int t^{\mp\frac{1}{2}-1}z\,dt(1+t)^{\pm\frac{1}{2}},$$

vnde duplex constructio:

altera $\frac{dz}{ds}=\frac{-2A+z(2-s)}{2s(1+s)}$ et $y=Cx\int\frac{z\,dt\sqrt{(1+t)}}{t\sqrt{t}}$

altera $\frac{dz}{ds}=\frac{-2A+z(2+s)}{2s(1+s)}$ et $y=Cx\int\frac{z\,dt}{\sqrt{t}(1+t)}$.

8.) Si $\lambda=-1$, habebitur:

$m=2$; $b=2\pm 1$; $n=-2$; $k=-2$; $\mu=2$; $\eta=2\mp 1$; $\nu=-2$; $\theta=0$

at

at posito $\frac{t}{xx}=s$, quaeri debet z ex hac aequatione:

$$\frac{dz}{ds}=\frac{\pm 2A - z(2+(2\pm 1)s)}{2s(1+s)},$$

vt posito $s=0$ fiat $z=A$ quo facto erit

$$y=\frac{C}{x}\int t^{\mp\frac{1}{2}}z\,dt(1+t)^{-1\pm\frac{1}{2}}$$

sicque duplex oritur constructio:

altera $\frac{dz}{ds}=\frac{2A-z(2+3s)}{2s(1+s)}$ et $y=\frac{C}{x}\int\frac{z\,dt}{\sqrt{t(1+t)}}$

altera $\frac{dz}{ds}=\frac{2A-z(2+s)}{2s(1+s)}$ et $y=\frac{C}{x}\int\frac{z\,dt\sqrt{t}}{(1+t)\sqrt{(1+t)}}$.

Omnino ergo sedecim constructiones sumus consecuti.

Scholion.

1076. Periculum faciamus ostendendi, quomodo hae constructiones, quae magis arduae videntur, aequationi propositae satisfaciant; atque in hunc finem eligamus constructionem posteriorem No. 4. quae habet

$$dz+\frac{z\,ds(2+3s)}{2s(1+s)}=\frac{A\,ds}{s(1+s)}$$

haec per $s\sqrt{(1+s)}$ multiplicata praebet integrale:

$$sz\sqrt{(1+s)}=A\int\frac{ds}{\sqrt{(1+s)}}=2A\sqrt{(1+s)}+B$$

seu $z=\frac{2A}{s}+\frac{B}{s\sqrt{(1+s)}}$.

Iam vt posito $s=0$ fiat $z=A$ debet esse $B=-2A$, vt sit

$$z=\frac{2A(\sqrt{(1+s)}-1)}{s\sqrt{(1+s)}}=\frac{2A}{txx}-\frac{2A}{txx\sqrt{(1+txx)}}.$$

Hinc fit

$$\left(\frac{dz}{dx}\right) = \frac{-4A}{tx^3} + \frac{2A(2+3txx)}{tx^3(1+txx)^{\frac{3}{2}}} \text{ et}$$

$$\left(\frac{ddz}{dx^2}\right) = \frac{12A}{tx^4} - \frac{6A(2+5txx+4ttx^4)}{tx^4(1+txx)^{\frac{5}{2}}}.$$

Cum nunc fit $y = C\int \frac{xxz\,dt}{\sqrt{t(1+t)}}$ erit

$$\left(\frac{dy}{dx}\right) = 2C\int \frac{xz\,dt}{\sqrt{t(1+t)}} + C\int \frac{xx\,dt}{\sqrt{t(1+t)}} \cdot \left(\frac{dz}{dx}\right) \text{ et}$$

$$\left(\frac{ddy}{dx^2}\right) = 2C\int \frac{z\,dt}{\sqrt{t(1+t)}} + 4C\int \frac{x\,dt}{\sqrt{t(1+t)}}\left(\frac{dz}{dx}\right) + C\int \frac{xx\,dt}{\sqrt{t(1+t)}}\left(\frac{ddz}{dx^2}\right)$$

hincque

$$x(1-xx)\frac{ddy}{dx^2} - x(1+xx)\frac{dy}{dx} + xxy =$$

$$C\int \frac{dt}{\sqrt{t(1+t)}} \cdot \left(\frac{2Axx}{t} - \frac{2Axx(1+4txx+3ttxx)}{t(1+txx)^{\frac{5}{2}}}\right)$$

quod integrale est

$$\frac{-4ACxx\sqrt{(1+t)}}{\sqrt{t}} + \frac{4ACxx\sqrt{(1+t)}}{(1+txx)^{\frac{3}{2}}\sqrt{t}}$$

$$= \frac{4ACxx\sqrt{(1+t)}}{\sqrt{t}}\left(\frac{1}{(1+txx)^{\frac{3}{2}}} - 1\right)$$

et hac forma exprimi poteſt:

$$-2Cx^4\left(3z + x\left(\frac{dz}{dx}\right)\right)\sqrt{t(1+t)}$$

vel etiam hoc modo:

$$-2Cx^4\left(\frac{2A+z}{1+txx}\right)\sqrt{t(1+t)}.$$

Expreſſio

Expreſſio autem iſta fit $=0$ primo ſi $t=-1$, deinde etiam ſi $t=0$, vnde valor pro y inuentus:

$$y=D\int\frac{dt}{t\sqrt{t(1+t)}}\left(1-\frac{1}{\sqrt{(1+txx)}}\right)$$

ita per integrationem definiri debet, vt euaneſcat poſito $t=0$; tum vero ponatur $t=-1$. Vel poſito $t=-v$ erit

$$y=D\int\frac{dv}{v\sqrt{v(1-v)}}\left(1-\frac{1}{\sqrt{(1-vxx)}}\right)$$

integrali ita ſumto vt euaneſcat poſito $v=0$, tum vero facto $v=1$.

Exemplum hoc ſufficit ad oſtendum quomodo conſtructiones exhibitae aequationi differentio-differentiali ſatisfaciant; interim vero ſi quantitas z tranſcendenter per logarithmos ſcilicet exprimitur, conſenſum nonniſi per calculos nimium taedioſos declarare licet.

Problema 136.

1077. Poſito $y=C\int(1+t)^{\nu-1}(a+tx)^{\lambda}dt$, in qua integratione quantitas x vt conſtans ſpectatur, integrali per terminos deinceps inueſtigandos definito, vt y aequetur certae functioni ipſius x, inuenire aequationes differentio-differentiales formae

$$Lxx.\frac{ddy}{dx^2}+Mx.\frac{dy}{dx}+Ny=0$$

cui ea functio ſatisfaciat.

Solutio.

Solutio.

Cum sit ex principiis ante stabilitis:

$$\frac{dy}{dx} = C\int \lambda t(1+t)^{\nu-1}(a+tx)^{\lambda-1}dt \quad \text{et}$$

$$\frac{ddy}{dx^2} = C\int \lambda(\lambda-1)tt(1+t)^{\nu-1}(a+tx)^{\lambda-2}dt$$

erit

$$Lxx.\frac{ddy}{dx^2} + Mx.\frac{dy}{dx} + Ny = C\int(1+t)^{\nu-1}(a+tx)^{\lambda-2}dt \times$$
$$(\lambda(\lambda-1)Lttxx + \lambda Mtx(a+tx) + N(a+tx)^2) =$$

$$C\int(1+t)^{\nu-1}(a+tx)^{\lambda-2}dt \left\{\begin{array}{l} Naa + 2Natx + Nttxx \\ \quad + \lambda Matx + \lambda Mttxx \\ \qquad + \lambda(\lambda-1)Lttxx \end{array}\right\}$$

quae formula sumta x constante absolute integrabilis esse debet. Ponatur ergo integrale

$$C(1+t)^{\nu}(a+tx)^{\lambda-1}(Paa+Qatx)$$

denotantibus P et Q functionibus quibuscunque ipsius x, erit eius differentiale:

$$C(1+t)^{\nu-1}(a+tx)^{\lambda-2}dt \times$$
$$(\nu(a+tx)(Paa+Qatx) + (\lambda-1)x(1+t)(Paa+Qatx) + Qax(1+t)(a+tx))$$

$$= C(1+t)^{\nu-1}(a+tx)^{\lambda-2}dt \left\{\begin{array}{lll} \nu Pa^3 & +\nu Qaatx & +\nu Qattxx \\ +(\lambda-1)Paax & +\nu Paatx & +(\lambda-1)Qattxx \\ +Qaax & +(\lambda-1)Paatx & +Qattxx \\ & +(\lambda-1)Qatxx & \\ & +Qaatx & \\ & +Qatxx & \end{array}\right\}$$

qua

qua forma cum illa comparata adipiscimur:

$$N = \nu P a + (\lambda - 1) P x + Q x$$

$$2N + \lambda M = (\nu + 1) Q a + (\lambda + \nu - 1) P a + \lambda Q x$$

$$N + \lambda M + \lambda(\lambda - 1) L = (\lambda + \nu) Q a$$

quarum aequationum extremae demta media praebent:

$$\lambda(\lambda - 1) L = -(\lambda - 1) P a + (\lambda - 1) P x + (\lambda - 1) Q a - (\lambda - 1) Q x$$

hincque

$$\lambda L = (a - x)(Q - P) \quad \text{seu} \quad L = \tfrac{1}{\lambda}(a - x)(Q - P)$$

secunda autem demto primae duplo dat:

$$\lambda M = (\lambda - \nu - 1) P a - 2(\lambda - 1) P x + (\nu + 1) Q a + (\lambda - 2) Q x$$

seu $\lambda M = ((\nu + 1) a + (\lambda - 2) x)(Q - P) + \lambda(a - x) P$.

Quare sumtis pro P et Q functionibus quibuscunque ipsius x, si functiones L, M, N ita definiantur vt sit

$$L = \tfrac{1}{\lambda}(a - x)(Q - P)$$

$$M = \tfrac{1}{\lambda}((\nu + 1) a + (\lambda - 2) x)(Q - P) + (a - x) P$$

$$N = x(Q - P) + (\nu a + \lambda x) P$$

aequationi differentio-differentiali

$$L x x d d y + M x\, d x\, d y + N y\, d x^2 = 0$$

satisfaciet formula integralis

$$y = C \int (1 + t)^{\nu - 1} (a + t x)^{\lambda} d t$$

tractata x vt constante, dummodo integrationis termini ita constituantur, vt vtroque haec expressio:

$$(1+t)^{\nu}(a+tx)^{\lambda-1}(Pa+Qtx) \text{ euanescat.}$$

Notari autem oportet hos terminos non ab x pendere debere. Primo autem patet hanc expressionem fieri $=0$ casu $t=-1$, si modo sit $\nu>0$. Deinde posito $t=\infty$ etiam euanescet, si modo sit $\nu+\lambda-1+1$ numerus negatiuus seu $\nu+\lambda<0$. Quocirca si sit $\nu>0$ et $\nu+\lambda<0$, integrale

$$y=C\int(1+t)^{\nu-1}(a+tx)^{\lambda}dt$$

ita capi debet, vt posito $t=-1$ euanescat, tum vero statuatur $t=\infty$, functioque ipsius x pro y resultans satisfaciet aequationi propositae.

Coroll. 1.

1078. Quoniam functiones P et Q in formulam integralem pro y assumtam, non ingrediuntur, manifestum est eandem formulam satisfacere omnibus aequationibus differentio-differentialibus, quicunque valores litteris P et Q tribuantur.

Coroll. 2.

1079. Sumto ergo $Q=P$ eadem formula integralis

$$y=C\int(1+t)^{\nu-1}(a+t)^{\lambda}dt$$

satis-

ſatisfacit etiam huic aequationi differentiali primi gradus

$$(a-x)xdy+(\nu a+\lambda x)ydx=0.$$

Huius vero integrale eſt

$$y=\frac{D(a-x)^{\lambda+\nu}}{x^{\nu}},$$

qui ergo valor quoque in genere noſtrae aequationi differentio-differentiali ſatisfacit, id quod tentanti mox patebit.

Coroll. 3.

1080. Hic ergo valor integralis

$$y=C\int(1+t)^{\nu-1}(a+tx)^{\lambda}dt$$

ſecundum terminos definitos ſumtus congruere debet cum formula algebraica $y=\frac{D(a-x)^{\lambda+\nu}}{x^{\nu}}$, ſi modo ſit $\nu>0$ et $\lambda+\nu<0$.

Scholion.

1081. Parum ergo integratio hoc problemate exhibita habet in receſſu. Verum reductio formulae integralis

$$y=C\int(1+t)^{\nu-1}(a+tx)^{\lambda}dt \text{ ad } y=\frac{D(a-x)^{\lambda+\nu}}{x^{\nu}}$$

eo magis eſt notatu digna, ad quam illa reducitur, ſi integrali ita ſumto vt euaneſcat poſito $t=-1$,

 pona-

ponatur $t = \infty$. Posito ergo $\lambda + \nu = -\mu$, vt μ et ν fiut numeri positiui, erit

$$C\int \frac{(1+t)^{\nu-1}dt}{(a+tx)^{\mu+\nu}} = \frac{D}{x^{\nu}(a-x)^{\mu}}.$$

Vel ponatur $1+t=z$ erit

$$C\int \frac{z^{\nu-1}dz}{(a-x+xz)^{\mu+\nu}} = \frac{D}{x^{\nu}(n-x)^{\mu}},$$

terminis illius integrationis existentibus $z=0$ et $z=\infty$. Verum etiam haec obseruatio non magni est momenti, nam posito $a-x=ux$, fit

$$\frac{C}{x^{\mu+\nu}}\int \frac{z^{\nu-1}dz}{(u+z)^{\mu+\nu}} = \frac{D}{x^{\mu+\nu}u^{\mu}},$$

ideoque haec formula $\int \frac{z^{\nu-1}dz}{(u+z)^{\mu+\nu}}$ ita integrata vt euanescat posito $z=0$ si tum ponatur $z=\infty$ hanc induet formam $\frac{A}{u^{\mu}}$, in qua A quantitatem constantem denotat ab u non pendentem. Pendet autem ab exponentibus μ et ν, lege ex casibus facile obseruanda. Scilicet posito

$$\int \frac{z^{\nu-1}dz}{(u+z)^{\mu+\nu}} = \frac{A}{u^{\mu}}$$

si fit $\nu=1$, integrale illud praebet $-\frac{1}{\mu(u+z)^{\mu}}+\frac{1}{\mu u^{\mu}}$, et

et pofito $z=\infty$, prodit $\frac{1}{\mu u^{\mu}}$, ita vt hoc cafu fit $A=\frac{1}{\mu}$. Si fit $\nu=2$, integratio quoque fuccedit, reperiturque $A=\frac{1}{\mu(\mu+1)}$ fi $\nu=3$ fit $A=\frac{1.2}{\mu(\mu+1)(\mu+2)}$, et fi $\nu=4$ fit $A=\frac{1.2.3}{\mu(\mu+1)(\mu+2)(\mu+3)}$, vnde in genere concludimus fore

$$A=\frac{1.2.3\ldots\ldots\ldots\ldots(\nu-1)}{\mu(\mu+1)(\mu+2)\ldots\ldots(\mu+\nu-1)}.$$

Quare integratione fecundum regulam praefcriptam inftituta erit

$$\frac{1}{\mu+1}\cdot\frac{2}{\mu+2}\cdot\frac{3}{\mu+3}\cdots\cdots\frac{\nu-1}{\mu+\nu-1}=\mu u^{\mu}\int\frac{z^{\nu-1}dz}{(u+z)^{\mu+\nu}}.$$

Quod fi exponens ν non fuerit integer, valor ipfius A ope interpolationis huius formnlae per factores procedentis definietur. Quadratura fcilicet circuli ingredietur fi exponens ν fractionem $\frac{1}{2}$ inuoluat, de huiusmodi autem interpolationibus alibi fufius egimus, neque hic locus eft hoc argumentum vberius profequendi. Reftat vltimum huius fectionis caput, quo aequationum differentio-differentialium integratio per approximationes docebitur.

CAPVT XII.

DE AEQVATIONVM DIFFERENTIO-DIFFERENTIALIVM INTEGRATIONE PER APPROXIMATIONES.

Problema 137.

1082.

Proposita aequatione differentio - differentiali quacunque, principia explicare, ex quibus integrationem per approximationes peti oportet.

Solutio.

Versetur aequatio proposita inter binas variabiles x et y, ac posito $dy = pdx$ et $dp = qdx$, dabitur aequatio inter quatuor quantitates x, y, p et q, ex qua q ita definire licebit, vt q aequetur functioni cuidam trium quantitatum x, y et p, qua vocata $= V$ sit $q = V$ seu $dp = Vdx$. Hic primo obseruandum est integrationem, vt sit determinata, duplicem determinationem requirere, seu duas conditiones quasi pro lubitu praescribi posse, quibus satisfaciat. Scilicet non sufficit, vt posito $x = a$ fiat $y = b$, quemadmodum in aequationibus differentialibus primi gra-

dus

dus vſu veniri vidimus, ſed aliam inſuper conditionem adiicere licet, quae ſit vt poſito $x=a$ fiat etiam $p=\frac{dy}{dx}=c$ quantitati datae. His ergo determinationibus conſtitutis, vt poſito $x=a$, fiat $y=b$ et $p=c$, totum integrationis negotium huc reducitur, vt ipſi x alium quemcunque valorem tribuendo, inueſtigentur valores reſpondentes ipſius y et ipſius p, hoc enim ſi praeſtiterimus, aequationis propoſitae integrale perfecte definiuerimus, vt nihil praeterea deſiderari poſſit. Quod cum in genere fieri nequeat, approximationis ratio in hoc conſiſtit, vt ipſi x valor quam minimum ab a diſcrepans tribuatur, qui ſit $x=a+\omega$, et inquiratur, quantum valores quantitatum y et p a primitiuis b et c ſint diſcrepaturi. Hic pro principio aſſumimus, dum x ab a ad $a+\omega$ increſcit, etiam quantitatum y et p valores tam parum mutatum iri, vt inde functio V nullam variationem notabilem patiatur. Quare ſi ponamus, ſtatuendo $x=a$, $y=b$ et $p=c$, fieri $V=F$, eundem valorem F quantitas V retinere cenſebitur, dum x ab a vsque ad $a+\omega$ augetur. Cum igitur pro hoc interuallo minimo habeamus $dp=Fdx$ erit integrando $p=Fx+\text{Conſt.}$ verum, quia poſito $x=a$ fieri debet $p=c$ erit $p=c+Fx-Fa$. Sit nunc $x=a+\omega$, atque habebimus $p=c+F\omega$, qui eſt valor ipſius p, valori $x=a+\omega$ reſpondens. Denique pro hoc minimo interuallo erit $dy=cdx$, ideoque $y=b+cx-ac$, et pro valore $x=a+\omega$ fit $y=b+c\omega$ qui eſt valor ipſius y valori $x=a+\omega$

con-

conueniens. Quocirca ſi valores primitiui ſint $x=a$, $y=b$ et $p=\frac{dy}{dx}=c$, ex iisque fiat $V=F$, ſequentes valores interuallo quam minimo ab illis remoti erunt:

$$x=a+\omega;\ y=b+c\omega;\ p=c+F\omega$$

qui ſi porro vt primitiui ſpectentur, ex iis ſimili modo per interuallum quam minimum progredi licet, ſicque tandem progreſſus per interuallum quantumuis magnum innoteſcet.

Coroll. 1.

1083. Quo minora capiantur haec interualla, eo minus a vero aberrabitur, dummodo quantitates c et F non ſint nimis magnae, ſin autem ea adeo in infinitum excreſcant, manifeſtum eſt errorem in quantitatibus y et p inſignem commiſſum iri.

Coroll. 2.

1084. Si quantitas c vel F fiat vehementer magna, interuallum quo y vel p creſcit pro dato accipi poteſt; ita poſito $c\omega=\psi$ erunt ſequentes valores $x=a+\frac{\psi}{c}$, $y=b+\psi$ et $p=c+\frac{F\psi}{c}$. At ſi F prodeat quantitas permagna, valor ipſius p intervallo minimo Φ augeri ſumatur, vt ſit $F\omega=\Phi$, eruntque valores ſequentes $x=a+\frac{\Phi}{F}$, $y=b+\frac{c\Phi}{F}$ et $p=c+\Phi$.

Coroll. 3.

Coroll. 3.

1085 Si b ſit quantitas infinita, pro valore proximo ipſius y expediet definiri

$$\frac{1}{y} = \frac{1}{b + c\omega} = \frac{1}{b} - \frac{c\omega}{bb} = \frac{-c\omega}{bb},$$

quae expreſſio vt ſit finita, etiam quantitas c infinita ſit oportet; alioquin valor ipſius y reſpondens non ſolum ipſi $x = a$ ſed etiam ipſi $x = a + \omega$ maneret infinitus.

Scholion 1.

1086. Quoties ſolutio alicuius problematis pendet ab integratione cuiuspiam aequationis differentialis ſecundi gradus, toties conditiones problematis binas determinationes ſuppeditare ſolent; quarum altera, dum ipſi x certus quidam valor $x = a$ tribuitur, exigit vt y datum valorem $y = b$, altera vero vt etiam ratio $\frac{dy}{dx} = p$ datum valorem $p = c$ conſequatur. Quodſi ergo in genere aequationem quandam differentio-differentialem integrare velimus, integrationem ita inſtituere licet, vt poſito $x = a$, fiat $y = b$ et $p = c$, quantitatibus a, b, c ab arbitrio noſtro pendentibus. Interim tamen quandoque vſu venire poteſt, vt poſito $x = a$, valores ipſarum y et p non penitus ab arbitrio noſtro pendeant, ſed ex natura aequationis iam datos valores ſortiantur, quibus caſibus defectus determinationis aliis conditionibus compenſatur. Veluti ſi proponatur haec aequatio:

$$xx(a-bx)ddy - 2x(2a-bx)dxdy + 2(3a-bx)ydx^2 = 6aadx^2$$

 quo-

quomodocunque ea per integrationem determinetur, posito $x=0$ necessario fit $y=a$ et $\frac{dy}{dx}=p=b$, ita vt pro casu $x=0$ valores quantitatum y et p minime arbitrio nostro relinquantur. Integrale autem completum reperitur $y=a+bx+\frac{(A+Bx)xx}{a-bx}$, vbi etiamsi constantes A et B pro lubitu assumantur, tamen semper posito $x=0$ prodit $y=a$ et $p=b$. Huiusmodi ergo casibus mirum non est, si pro dato ipsius x valore, quantitum y et p valores arbitrio nostro haud permittantur.

Scholion 2.

1087. Exposita ratio aequationes differentio-differentiales per approximationes integrandi, dum per interualla minima progredimur, quemadmodum etiam in aequationibus differentialibus primi gradus fecimus, certis casibus difficultatibus inuoluitur, vt nisi remedium afferatur, in vsum vocari nequeat. Primum hoc euenit, quando $c=\infty$, tum enim quantumuis exiguum accipiatur interuallum ω, neque ipsius y neque ipsius p valorem cognoscere licet. Simile quoque incommodum turbat, si positis $x=a$, $y=b$ et $p=c$ functio V fiat infinita, ideoque prodeat $F=\infty$, quo casu valor ipsius p non definitur. Deinde etiam casus quibus vel c vel F euanescit, seorsim tractari conuenit: etsi enim tum valores ipsarum y et p satis accurate ostenduntur, tamen quia nullam mutationem patiuntur, dum mutatio

tatio altiore ipfius ω poteftate exprimitur, hanc ipfam mutationem inueftigari vtile eft, quo in progreffu minus a veritate aberretur. Sin autem quantitas b euadat infinita, iam animaduertimus, loco ipfius y eius reciprocum $\frac{1}{y}$ explorari debere. Quemadmodum ergo difficultatibus ante memoratis fit occurrendum, diligentius perpendamus.

Problema 138.

1088. Si integrationem per interualla inftituendo, pro initio cuiuspiam interualli pofito $x=a$, $y=b$ et $p=c$ eueniat, vt quantitas c fit vel euanefcens vel infinitia, integrationem per hoc interuallum abfoluere.

Solutio.

Praecedens approximatio dederat $y=b+c(x-a)$; vnde fi $c=0$, incrementum ipfius y altiore poteftate ipfius $x-a$ exprimetur, fcilicet $y=b+A(x-a)^{\lambda}$ exiftente $\lambda>1$, reiectisque altioribus poteftatibus, quae prae hac ob interuallum $x-a$ minimum recte contemnuntur. Sin autem fit $c=\infty$ valor ipfius y fimili modo repraefentari poteft, $y=b+A(x-a)^{\lambda}$ exiftente $\lambda<1$, ita tamen vt nihilo fit maior; vtroque ergo cafu eadem inueftigatio eft, vt ex aequatione propofita $dp=Vdx$ tam coefficiens A quam exponens λ definiatur. Iam ex illa aequatione deducimus

$\frac{dy}{dx}=p=\lambda A(x-a)^{\lambda-1}$ et $dp=\lambda(\lambda-1)A(x-a)^{\lambda-2}dx$;

 ac

ac necesse est eandem expressionem resultare, si in formula Vdx statuatur

$$y=b+A(x-a)^{\lambda} \text{ et } p=\lambda A(-a)^{\lambda-1};$$

vnde euidens est, fore

$$c=0, \text{ si } \lambda>1 \text{ et } c=\infty, \text{ si } \lambda<1.$$

Cum iam V sit functio ipsarum x, y et p, ponatur vbique $x=a$ nisi quatenus formula $x-a$ inest, quae relinquatur, tum vero $y=b$, nisi hinc prodeat $V=0$, vel $V=\infty$; hoc enim si eueniat pro y valor $b+A(x-a)^{\lambda}$ scribatur, similique modo pro p scribatur $\lambda A(x-a)^{\lambda-1}$. Reiiciantur autem formulae $x-a$ potestates altiores prae inferioribus, sicque pro V orietur expressio huius formae $C(x-a)^{\mu}$, quae formulae $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ aequari debet, vnde tam coefficiens assumtus A quam exponens λ definietur, ideoque vero proximi valores

$$y=b+A(x-a)^{\lambda} \text{ et } p=\lambda A(x-a)^{\lambda-1}$$

innotescent, qui eo minus a veritate recedent, quo minor differentia inter a et x constituatur. Casus autem quo $\lambda=1$ per se est perspicuus, atque in praecedente problemate pertractatus, cum is sit solus quo quantitas c finitum nanciscitur valorem.

Coroll. 1.

1089. Si eueniat vt posito $c=\infty$, quo casu esse debebat $\lambda<1$, functio V finitum obtineat valorem, cui formula $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ aequari nequit

nequit, casus per se nihil habet difficultatis, et valor ipsius y etiam pro minimo excessu ipsius x super a reuera infinitus euadet.

Coroll. 2.

1090. Facilius hoc perspicere licet ex exemplo $dp = 6xdx$ vnde fit

$$p = c + 3xx - 3aa = \frac{dy}{dx} \text{ hincque}$$

$$y = b + (c - 3aa)(x - a) + x^3 - a^3 \text{ seu}$$

$$y = b - ac + 2a^3 + (c - 3aa)x + x^3.$$

Si ergo constans c sumatur infinita, valor ipsius y semper erit infinitus solo excepto casu $x = a$.

Coroll. 3.

1091. Sin autem sumto $c = 0$, quo casu esse debet $\lambda > 1$, functio V finitum habeat valorem eumque adeo constantem, posito $x = a$ et $y = b$, ei formula $\lambda(\lambda - 1)A(x - a)^{\lambda - 2}$ aequabitur sumendo $\lambda = 2$, et $2A =$ illi valori constanti. Veluti in praecedente exemplo fit $V = 6a = 2A$, hinc $A = 3a$, eritque proxime $y = b + 3a(x - a)^2$; quod etiam congruit cum integrali inuento, quod posito $c = 0$ est

$$y = b + 2a^3 - 3aax + x^3 = b + (x + 2a)(x - a)^2,$$

quae expressio facto $x = a$ in illam abit.

Scholion 1.

1092. Posito $c=0$ functio V si in ea scribatur $x=a$, $y=b$ et $p=c=0$, valorem nanciscetur vel infinite magnum, vel finitum, vel adeo euanescentem. Primo casu, quo fit $V=\infty$, vt ei aequari possit $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ sumto $x=a$, necesse est sit $\lambda<2$, existente $\lambda>1$; vt autem hinc quantitates A et λ definiantur, in functione V scribi oportet

$$y=b+A(x-a)^{\lambda} \text{ et } p=\lambda A(x-a)^{\lambda-1},$$

itemque $x=a$, nisi vbi formula $x-a$ occurrit; hoc modo quia per hypothesin casu $x=a$ fit $V=\infty$, iste valor prodibit $=C(x-a)^{-\alpha}$ quo collato cum $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ reperientur A et λ dum ne prodeat $\lambda<1$, qui casus cum $c=0$ subsistere nequit. Secundo casu quo prodit $V=$ quantitati finitae, capi oportet $\lambda=2$, sin autem tertio casu fit $V=0$, sumi debet $\lambda>2$, vt eius valor in formula $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ contineatur. At si debeat esse $c=\infty$, fieri nequit vt vidimus, vt functio V finitum obtineat valorem, multo minus euanescentem, nisi quidem casus incongruos, quibus y perpetuo maneat infinita, admittere velimus. Tum igitur functio V necessario valorem infinitum induit, cum formula $\lambda(\lambda-1)A(x-a)^{\lambda-2}$ comparandum ita vt sit $\lambda<1$. Hinc igitur patet, determinationem quantitatis c non semper arbitrio nostro relinqui, sed quandoque ex indole ipsius aequationis nobis praescri-

ſcribi. Veluti ſi proponatur haec aequatio $ddy = \frac{2dx^2}{(x-a)^3}$ erit $\frac{dy}{dx} = p = A - \frac{1}{(x-a)^2}$ et $y = B + Ax + \frac{1}{x-a}$ vnde poſito $x = a$ fit $c = A - \frac{1}{o^2}$ et $b = B + Aa + \frac{1}{o}$ ergo $A = c + \frac{1}{o^2}$ et $B = b - ac - \frac{a}{o^2} + \frac{1}{o}$, quare ne aequatio integralis omnino in infinitis verſetur, litterae b et c non poſſunt non eſſe infinitae.

Scholion 2.

1093. Non autem omnes ordines infinitorum et euaneſcentium in formula $(x-a)^\lambda$ caſu $x = a$, contineri iam obſeruauimus; expreſſio ſcilicet $xxlx$ caſu $x = 0$, infinities ſuperat poteſtatem ſecundam x^2, interim tamen infinities minor eſt poteſtate $x^{2-\alpha}$, quantumuis etiam exiguae fractio α accipiatur. Quare ſi in ſuperiori ſolutione formulas ita inſtruere velimus, vt ad omnes ordines tam infinitorum quam euaneſcentium pateant, ſtatui conueniet:

$$y = b + A(x-a)^\lambda (l(x-a))^\mu,$$

vnde fit

$$\frac{dy}{dx} = p = \lambda A(x-a)^{\lambda-1}(l(x-a))^\mu + \mu A(x-a)^{\lambda-1}(l(x-a))^{\mu-1};$$

at poſito $x = a$ pars prior eſt ad poſteriorem vt $l(x-a)$ ad 1, hoc eſt vt $\infty : 1$ ex quo ſufficit ſumſiſſe

$$p = \lambda A(x-a)^{\lambda-1}(l(x-a))^\mu;$$

ex quo ſimili modo colligitur

$$\frac{dp}{dx} = \lambda(\lambda-1)A(x-a)^{\lambda-2}(l(x-a))^\mu,$$

quae

quae expreſſio cum functione V poſtquam in ea ſcripſerimus $x=a$, $y=c$ et $p=c$, ſeu potius

$$y=b+A(x-a)^{\lambda}(l(x-a))^{\mu} \text{ et } p=\lambda A(x-a)^{\lambda-1}(l(x-a))^{\mu}$$

comparari debet, vt inde tam conſtans A quam exponentes λ et μ innoteſcant. Haec ergo tenenda ſunt in iſta inueſtigatione, quo ea ad plures caſus extendatur.

Problema 139.

1094. Approximationem ante expoſitam accuratius perſequi, vt ſumtis interuallis etiam paulo maioribus minus a vero aberretur.

Solutio.

Poſito $dy=pdx$ aequatio differentio-differentialis hac forma $\frac{dp}{dx}=V$ exhibeatur, ex qua ante p ita definiuimus quaſi V eſſet quantitas conſtans, pro interuallo ſaltem vehementer paruo, vnde obtinuimus $p=c+V(x-a)$ poſtquam ſcilicet in V poſuerimus $x=a$, et $y=b$, et $p=c$ qui ſunt valores primitiui per interuallum $x-a=\omega$ retinendi. Cum autem functio V interea non ſit conſtans quia x, y et p inuoluit, reuera erit

$$p=c+V(x-a)-\int(x-a)dV.$$

Ponamus igitur

$$dV=Pdx+Qdy+Rdp$$

vt ſit

$$dV=(P+Qp+RV)dx,$$

et

et nunc quantitatem $P+Qp+RV$ vt constantem spectemus, cuius valor prodeat ponendo $x=a$, $y=b$ et $p=c$, quo facto V in F abire supra sumsimus, eritque

$$p=c+F(x-a)-\tfrac{1}{2}(P+Qc+RF)(x-a)^2.$$

Hinc porro ob $dy=pdx$ fit

$$y=b+c(x-a)+\tfrac{1}{2}F(x-a)^2-\tfrac{1}{6}(P+Qc+RF)(x-a)^3$$

similique modo approximationem vlterius prosequi licet. Quando autem quantitates P, Q, R et V formulam $x-a$ eiusue potestates complectuntur, quam non amplius vt constantem spectare licet, eius ratio in integratione est habenda, qua fit vt in seriebus approximantibus formulae $x-a$ potestates non ordine ascendant. Tum igitur conueniet pro p eiusmodi seriei initium assumi:

$$p=c+A(x-a)^\lambda, \text{ vnde fit}$$

$$y=b+c(x-a)+\frac{A}{\lambda+1}(x-a)^{\lambda+1} \text{ et quia}$$

$$\frac{dp}{dx}=\lambda A(x-a)^{\lambda-1},$$

huic formulae aequari debet functio V, postquam in ea pro y et p valores assumtos et a pro x, scripserimus, nisi formula $x-a$ ingrediatur, hoc modo tam exponens λ quam coefficiens A determinabitur.

Si c sit $=0$ vel $=\infty$, eius ratio potest in calculum introduci vt ponatur

$$p=f(x-a)^n+A(x-a)^\lambda,$$

vnde fit

$$y = b + \frac{f}{n+1}(x-a)^{n+1} + \frac{A}{\lambda+1}(x-a)^{\lambda+1},$$

qui valores si loco x et p substituantur in functione V prodire debet

$$nf(x-a)^{n-1} + \lambda A(x-a)^{\lambda-1}.$$

Coroll. 1.

1095. Hoc modo per interualla continuo vlterius progredi licet, dummodo singula non maiora accipiantur, quam vt errores commissi maneant insensibiles; atque hac quidem correctione errores illi diminuuntur, vt interualla etiam maiora statui queant.

Coroll. 2.

1096. Pro primo scilicet interuallo valores primitiui $x = a$, $y = b$ et $p = c$ pro lubitu assumuntur, et valores in fine interualli inuenti praebent valores initiales pro secundo interuallo, ex quibus calculus pro hoc interuallo perinde expeditur ac pro primo; sicque continuo vlterius est progrediendum.

Scholion.

1097. Huius problematis duplicem solutionem dedimus, quarum prior etsi latissime patere videtur, certis tamen casibus in vsum vocari nequit; iis ergo altera solutione vti conueniet. Existunt autem tantum

tantum plerumque pauciſſima eiusmodi interualla, quae poſteriorem methodum poſtulant, dum reliqua omnia ope prioris expedire licet. Euenit hoc quando pro quopiam interuallo quantitates V et c vel euaneſcunt vel in infinitum excreſcunt; quin etiam fieri poteſt, vt quantumuis exiguum interuallum accipiatur, quantitates y et p variationibus infinitis ſint obnoxiae, quarum repraeſentatio determinationem prorſus ſingularem requirit. Veluti ſi proponatur haec aequatio $ddy + \frac{ydx^2}{xx} = 0$, interuallum ab $x = 0$ vsque ad $x = \omega$, etiamſi ω quam minimum aſſumatur, infinitam mutationem in valoribus y et p indicat; id quod ex eius integrali completo perſpicitur, quod cum ſit

$$y = Ax^{\frac{1}{2}} \text{ſin.} \left(\frac{\sqrt{3}}{2} lx + \alpha\right), \text{ hincqne}$$

$$p = \frac{A}{2\sqrt{x}} \text{ſin.} \left(\frac{\sqrt{3}}{2} lx + \alpha\right) + \frac{A\sqrt{3}}{2\sqrt{x}} \text{coſ.} \left(\frac{\sqrt{3}}{2} lx + \alpha\right) \text{ ſeu}$$

$$p = \frac{A}{\sqrt{x}} \text{ſin.} \left(\frac{\sqrt{3}}{2} lx + \alpha + 60^\circ\right),$$

euidens eſt ſi $x = 0$, fore quidem $y = 0$, ſed ipſius p valorem eſſe incertum. At ipſi x valorem quam minimum tribuendo, y quidem minimum retinebit valorem, ſed qui pro minimo interuallo, modo ſit poſitiuus modo euaneſcens, modo negatiuus, ob maximam mutationem, quam lx patitur, quantitas autem p interea tranſit per omnes mutationes poſſibiles. Idem luculentius perſpicitur ex hoc exemplo

$$ddy + \frac{2dxdy}{x} - \frac{ffydx^2}{x^4} = 0$$

cuius

cuius integrale completum eſt $y = A \sin.(\frac{f}{x} + \alpha)$; dum enim x a o ad ω creſcit, angulus $\frac{f}{x} + \alpha$ ab infinito ad finitum tranſibit, eiusque ſinus interea omnes mutationes ab $+1$ ad -1 infinities adeo ſubiit. Quando ergo eiusmodi interualla occurrunt, mirum non eſt, ſi conſuetae methodi approximandi deficiant, quippe quae hoc principio innituntur, quod mutationes per interualla minima, ſint etiam valde paruae; his autem interuallis exceptis ſolutio praeſcripta ſemper cum vſu adhiberi poteſt.

Exemplum 1.

1098. *Propoſita aequatione* $ddx + \frac{y\,dx^2}{fx} = 0$, *eius integrationem per approximationem abſoluere.*

Cum ergo ſit $dp = -\frac{y\,dx}{fx}$, erit $V = \frac{-y}{fx}$; quare ſi pro initio interualli ſit $x = a$, $y = b$ et $p = c$, inde tantillum progrediendo, per ſolutionem priorem ob $P = \frac{y}{fxx} = \frac{b}{aaf}$, $Q = \frac{-1}{fx} = \frac{-1}{af}$ et $R = 0$, habebimus:

$$p = c - \frac{b}{af}(x-a) - \frac{1}{2}\left(\frac{b}{aaf} - \frac{c}{af}\right)(x-a)^2 \text{ et}$$
$$y = b + c(x-a) - \frac{b}{2af}(x-a)^2.$$

Sumto ergo $x - a = \omega$, pro interuallo ſequente erunt valores initiales:

$$a' = a + \omega;\; b' = b + c\omega - \frac{b\omega^2}{2af} \text{ et } c' = c - \frac{b\omega}{af} - \frac{(b-ac)\omega\omega}{2aaf}$$

vnde ſimili modo valores initiales pro interuallo ſequen-

fequente colliguntur. Verum fi pro quopiam intervallo fiat $a=0$, operatio peculiari modo inftitui debet. Pofito fcilicet pro initio huius interualli $x=0$, $y=b$ et $p=c$ ftatuatur

$$p=c+Ax^{\lambda} \text{ et } y=b+cx+\frac{Ax^{\lambda+1}}{\lambda+1} \text{ erit}$$

$$\frac{dp}{dx}=\lambda Ax^{\lambda-1}=\frac{-y}{fx}=\frac{-b}{fx}-\frac{c}{f}-\frac{Ax^{\lambda}}{(\lambda+1)f},$$

cui nifi fit $b=0$ fatisfieri nequit; prodiret enim $\lambda=0$ et $A=\infty$, vnde concludimus poni debere $y=b+Axlx$ vt fit $p=Alx+A$ et $\frac{dp}{dx}=\frac{A}{x}=\frac{-b-Axlx}{fx}$, hinc $A=-\frac{b}{f}$. Verum quo hinc p accuratius cognofcere liceat ftatuamus $y=b+Axlx+Bx$, erit

$$p=Alx+A+B \text{ et } \frac{dp}{dx}=\frac{A}{x};$$

vnde concluditur vt ante $A=\frac{-b}{f}$ et B manet indeterminatum: ita vt fit

$$y=b-\frac{bx}{f}lx+Bx \text{ et } p=\frac{-b}{f}lx-\frac{b}{f}+B$$

nifi ergo fit $b=0$, cafu $x=0$, quantitas c neceffario eft infinita. Quamobrem fi interualli initio fit $x=0$, $y=b$ et $p=\infty$, pro eius fine et initio fequentis erit $x=\omega$; $y=b-\frac{b\omega}{f}l\omega$ et $p=\frac{-b}{f}l\omega$.

Exemplum 2.

1099. *Propofita fit haec aequatio:*

$$xxddy-2xdxdy+2ydx^2=\frac{x^2ydx^2}{ff}$$

quam per approximationem integrari oporteat.

 Cum

Cum ſit

$$\frac{dp}{dx}=\frac{2p}{x}-\frac{2y}{xx}+\frac{y}{ff}=V\text{ , erit}$$

$$P=\frac{-2p}{xx}+\frac{4y}{x^3};\ Q=\frac{-2}{xx}+\frac{1}{ff}\text{ et }R=\frac{2}{x}.$$

Hinc ſi pro cuiusque interualli initio ſit $x=a, y=b$, $p=c$, ob $F=\frac{2c}{a}-\frac{2b}{aa}+\frac{b}{ff}$ erit:

$$p=c+\left(\frac{2c}{a}-\frac{2b}{aa}+\frac{b}{ff}\right)(x-a)-\tfrac{1}{2}\left(\frac{c}{ff}+\frac{2b}{aff}\right)(x-a)^2\text{ et}$$

$$y=b+c(x-a)+\tfrac{1}{2}\left(\frac{2c}{a}-\frac{2b}{aa}+\frac{b}{ff}\right)(x-a)^2$$

vnde calculus per interualla facile continuatur, dum ne ſit $a=0$. Hoc autem caſu quo $a=0$, difficulter interuallum computo definitur, quia tum fieri nequit, vt quantitatibus b et c dati valores tribuantur, id quod inde facillime intelligitur, quod aequationis propoſitae integrale completum eſt

$$y=Ae^{\frac{x}{f}}x+Be^{\frac{-x}{f}}x,$$

poſito enim $x=0$, neceſſario fit $y=0$, niſi coefficientes A et B infinitos capere velimus. Sumto autem $b=0$, approximatio eſt in promtu.

CALCVLI

CALCVLI INTEGRALIS
LIBER PRIOR.

PARS SECVNDA

SEV

METHODVS INVENIENDI FVNCTIONES

VNIVS VARIABILIS EX DATA RELATIONE DIFFERENTIALIVM SECVNDI ALTIORISVE GRADVS.

SECTIO POSTERIOR

DE

RESOLVTIONE AEQVATIONVM DIFFERENTIALIVM TERTII ALTIORVMQVE GRADVVM QVAE DVAS TANTVM VARIABILES INVOLVVNT.

CAPVT I.

DE
INTEGRATIONE FORMVLARVM
DIFFERENTIALIVM TERTII ALTIORISVE GRADVS SIMPLICIVM.

Problema 140.

1100.

Sumto elemento dx constante inuenire integrale completum harum formularum $d^3y=0$, $d^4y=0$, $d^5y=0$ etc. atque in genere huius $d^ny=0$.

Solutio.

Cum dx sit constans, aequatio $d^3y=0$ per integrationem dat $ddy=\alpha dx^2$, hincque porro integrando $dy=\alpha x dx+\beta dx$ et tandem $y=\frac{1}{2}\alpha x^2+\beta x+\gamma$.

Simili modo ex aequatione $d^4y=0$, per quadruplicem integrationem reperitur

1) $d^3y=\alpha dx^3$;

2) $ddy=\alpha x dx^2+\beta dx^2$;

3) $dy=\frac{1}{2}\alpha xxdx+\beta x dx+\gamma dx$ et tandem

4) $y=\frac{1}{6}\alpha x^3+\frac{1}{2}\beta x^2+\gamma x+\delta$.

Ex aequatione autem $d^5y=0$, integratio quinquies repetita dat:

$$y=\frac{1}{24}\alpha x^4+\frac{1}{6}\beta x^3+\frac{1}{2}\gamma x^2+\delta x+\varepsilon.$$

At huius aequationis $d^6y=0$ integrale colligitur:

$$y=\frac{1}{120}\alpha x^5+\frac{1}{24}\beta x^4+\frac{1}{6}\gamma x^3+\frac{1}{2}\delta x^2+\varepsilon x+\zeta$$

ficque ad huiusmodi formas $d^ny=0$, quanticumque fuerint gradus progredi licet, dummodo n fuerit numerus integer pofitiuus.

Coroll. 1.

1101. A fimpliciffima forma ergo incipiendo integralia fequenti ordine procedunt:

Formularum	Integralia completa funt
$dy=0$	$y=\alpha$
$ddy=0$	$y=\alpha x+\beta$
$d^3y=0$	$y=\frac{1}{2}\alpha x^2+\beta x+\gamma$
$d^4y=0$	$y=\frac{1}{6}\alpha x^3+\frac{1}{2}\beta x^2+\gamma x+\delta$
etc.	etc.

Coroll. 2.

Coroll. 2.

1102. Quia constantes α, β, γ etc. ab arbitrio nostro pendent, fractiones tuto reiicere licet, eritque

Formularum	Integralia
$dy=0$	$y=\alpha$
$ddy=0$	$y=\alpha x+\beta$
$d^3y=0$	$y=\alpha xx+\beta x+\gamma$
$d^4y=0$	$y=\alpha x^3+\beta x^2+\gamma x+\delta$
$d^5y=0$	$y=\alpha x^4+\beta x^3+\gamma x^2+\delta x+\epsilon$
etc.	etc.

Coroll. 3.

1103. Quoti ergo ordinis est formula differentialis, totidem constantes arbitrarias eius integrale completum complectitur, quas pro quouis casu oblato secundum conditiones praescriptas definiri oportet.

Scholion 1.

1104. Ponendo $dy=pdx$, $dp=qdx$, $dq=rdx$, $dr=sdx$ etc. omnes aequationes differentiales altiorum ordinum ad quantitates finitas reducuntur, in quibus nulla amplius ratio eius elementi, quod constans assumitur, habetur. Atque hinc formae omnium aequationum differentialium sequenti modo repraesentari possunt:

Aequationum differentialium	Forma generalis
I. gradus	$p = f:(x \text{ et } y)$
II. gradus	$q = f:(x, y, \text{et } p)$
III. gradus	$r = f:(x, y, p, \text{et } q)$
IV. gradus	$s = f:(x, y, p, q, \text{et } r)$
	etc.

vbi quantitates y, p, q, r, s etc. ita excludendo dx a se inuicem pendent, vt cum sit

$$dx = \frac{dy}{p} = \frac{dp}{q} = \frac{dq}{r} = \frac{dr}{s} = \frac{ds}{t} \text{ etc.}$$

sequentes relationes locum habeant:

$qdy = pdp$; $rdy = pdq$; $sdy = pdr$; $tdy = pds$ etc.

$rdp = qdq$; $sdp = qdr$; $tdp = qds$ etc.

$sdq = rdr$; $tdq = rds$ etc. $tdr = sds$ etc.

quarum formularum quaedam per se sunt integrabiles veluti:

$\int qdy = \frac{1}{2}pp$; $\int rdp = \frac{1}{2}qq$; $\int sdq = \frac{1}{2}rr$; $\int tdr = \frac{1}{2}ss$ etc.

ex quibus porro ob $\int zdv = vz - \int vdz$ istae concluduntur:

$\int ydq = yq - \frac{1}{2}pp$; $\int pdr = pr - \frac{1}{2}qq$; $\int qds = qs - \frac{1}{2}rr$; $\int rdt = rt - \frac{1}{2}ss$ etc.

quarum ope ex praecedentibus deducitur:

$\int sdy = pr - \frac{1}{2}qq$; hinc $\int yds = ys - pr + \frac{1}{2}qq$

$\int tdp = qs - \frac{1}{2}rr$; hinc $\int pdt = pt - qs + \frac{1}{2}rr$

$\int udq = rt - \frac{1}{2}ss$; hinc $\int qdu = qu - rt + \frac{1}{2}ss$.

Hinc

Hinc porro definitur $\int y\,du = yu - \int u\,dy$, at $\frac{dy}{p} = \frac{dt}{u}$, vnde $\int y\,du = yu - \int p\,dt = yu - pt + qs - \frac{1}{2}rr$.

Quare ſi differentialia iterum introducamus, obtinebimus ſequentes formulas integrales:

$$\int y\,dy = \tfrac{1}{2}yy$$
$$\int y\,d^3y = y\,ddy - \tfrac{1}{2}dy^2$$
$$\int y\,d^5y = y\,d^4y - dy\,d^3y + \tfrac{1}{2}ddy^2$$
$$\int y\,d^7y = y\,d^6y - dy\,d^5y + ddy\,d^4y - \tfrac{1}{2}d^3y^2$$

etc.

ita vt formula $\int y\,d^ny$ ſit integrabilis, quoties n eſt numerus impar.

Scholion 2.

1105. In aequationibus differentialibus ſecundi gradus formas ſimpliciores ita conſtituimus, vt q aequetur functioni vel ipſius x tantum, vel ipſius y vel ipſius p, quas litteras maiuſculas pro functionibus minuſcularum ſcribendo, ita repraeſentare licet, vt ſit

$$\text{vel } q = X \text{ vel } q = Y \text{ vel } q = P.$$

Hinc ſimili modo pro aequationibus differentialibus tertii gradus formas ſimpliciores conſtituere poſſumus:

$$r = X;\ r = Y;\ r = P;\ r = Q$$

ita vt tantum binas quantitates variabiles inuoluant.

Pro quarto autem gradu essent formae simpliciores:

$$s=X;\ s=Y;\ s=P;\ s=Q;\ s=R$$

et pro quinto:

$$t=X;\ t=Y;\ t=P;\ t=Q;\ t=R;\ t=S$$

atque ita porro pro superioribus.

Verum hae formae non omnes aeque integrationem admittunt, dum aliae ne semel quidem, aliae semel tantum, aliae per omnes integrationes vsque ad relationem inter x et y perduci possunt, cuiusmodi sunt primae quaeque in quouis gradu. Semper autem id est propositum, vt relatio inter binas variabiles principales x et y eliciatur.

Problema 141.

1106. Posito $dy=pdx$, $dp=qdx$, $dq=rdx$, $dr=sdx$, $ds=tdx$ etc. pro quouis differentialium gradu, si litterarum p, q, r, s, t etc. quaepiam aequetur functioni ipsius x, quae sit X, inuenire relationem inter x et y.

Solutio.

Si primo sit $p=X$, per dx multiplicando erit $pdx=dy=Xdx$, hincque $y=\int Xdx$, qui est casus formularum differentialium primi gradus simplicium.

Sit

Sit secundo $q = X$, erit $q\,dx = dp = X\,dx$, hinc $p = \int X\,dx$, et $p\,dx = dy = dx\int X\,dx$, ergo $y = \int dx\int X\,dx$ seu per simplicia integralia $y = x\int X\,dx - \int Xx\,dx$.

Sit tertio $r = X$ ob $dq = r\,dx$ erit $q = \int X\,dx$, hincque $p = \int q\,dx = \int dx\int X\,dx = x\int X\,dx - \int Xx\,dx$, ac tandem

$$y = \int p\,dx = \int dx\int dx\int X\,dx = \tfrac{1}{2}xx\int X\,dx - x\int Xx\,dx + \tfrac{1}{2}\int Xxx\,dx.$$

Sit quarto $s = X$ ac reperitur $y = \int dx\int dx\int dx\int X\,dx$, quae expressio euoluitur in hanc:

$$y = \tfrac{1}{6}x^3\int X\,dx - \tfrac{1}{2}xx\int Xx\,dx + \tfrac{1}{2}x\int Xxx\,dx - \tfrac{1}{6}\int Xx^3\,dx.$$

Sit quinto $t = X$ erit $y = \int dx\int dx\int dx\int dx\int X\,dx$ seu

$$y = \tfrac{1}{24}x^4\int X\,dx - \tfrac{1}{6}x^3\int Xx\,dx + \tfrac{1}{4}x^2\int Xxx\,dx - \tfrac{1}{6}x\int Xx^3\,dx + \tfrac{1}{24}\int Xx^4\,dx$$

vnde lex vlterius progrediendi est manifesta.

Coroll. 1.

1107. Tot ergo habentur formulae integrales, quoti gradus aequatio fuerit differentialis, et quia quaelibet constantem arbitrariam assumit, totidem constantes in integrale ingrediuntur, quibus id completum redditur; quod idem ex priori forma, vbi totidem signa integralia implicantur, intelligitur.

Coroll. 2.

1108. Sumto elemento dx constante, sequentium formularum more consueto expressarum integralia

gralia completa ita se habebunt:

I. Si $dy = Xdx$ est $y = \int Xdx$

II. Si $ddy = Xdx^2$ est $1y = x\int Xdx - \int Xxdx$

III. Si $d^3y = Xdx^3$ est

$$2y = x^2\int Xdx - 2x\int Xxdx + \int Xx^2dx$$

IV. Si $d^4y = Xdx^4$ est

$$6y = x^3\int Xdx - 3x^2\int Xxdx + 3x\int Xx^2dx - \int Xx^3dx$$

V. Si $d^5y = Xdx^5$ est

$$24y = x^4\int Xdx - 4x^3\int Xxdx + 6x^2\int Xx^2dx - 4x\int Xx^3dx + \int Xx^4dx$$

etc.

Scholion.

1109. Formulas autem, quas supra secundo loco constituimus, functionem Y complectentes, post secundum gradum integrare non licet. Ex tertio enim ordine formula $r = Y$ etsi novimus esse

$$r = \frac{pdq}{dy} = \frac{qdq}{dp} = \frac{dq}{dx},$$

nullo modo integrari potest; neque etiam hinc q per y determinari potest. Nam sumta forma $pdq = Ydy$ existente $pdp = qdy$ ob $p = \frac{Ydy}{dq}$, erit

$$dp = \frac{dydY}{dq} + Y\,d.\frac{dy}{dq},$$

hincque p elidendo:

$$\frac{Ydy^2dY}{dq^2} + \frac{YYdy}{dq}\,d.\frac{dy}{dq} = q\,dy,$$

quae

quae quidem aequatio est secundi gradus, sed neutiquam in genere resolutionem admittit. Ex quarto genere formula $s=Y$, ob $\int s\,dy = pr - \frac{1}{2}qq = \int Y\,dy$ semel integrari potest, sed hinc vlterius progredi non licet. Quas autem supra pro quouis gradu formulas simpliciores vltimo loco constituimus itemque penultimo, eae tractabiles deprehenduntur: earum ergo integrationem inuestigemus.

Problema 142.

1110. Posito vt hactenus $dy = p\,dx$, $dp = q\,dx$, $dq = r\,dx$ etc. litterae Y, P, Q, R denotent functiones cuiusque litterae minusculae cognominis, inuestigare integralia harum formularum simplicium

$$p = Y,\ q = P,\ r = Q,\ s = R,\ t = S \text{ etc.}$$

Solutio.

Aequatio prima ob $p = \frac{dy}{dx}$ statim dat $dx = \frac{dy}{Y}$ ideoque $x = \int \frac{dy}{Y}$.

Aequatio secunda $q = P$ ob $q = \frac{dp}{dx}$ praebet $dx = \frac{dp}{P}$ et $dy = \frac{p\,dp}{P}$, vnde cum P sit functio ipsius p, vtraque variabilis x et y per p determinatur hoc modo

$$x = \int \frac{dp}{P} \text{ et } y = \int \frac{p\,dp}{P}.$$

Aequatio tertia $r = Q$ ob $r = \frac{dq}{dx}$ dat $dx = \frac{dq}{Q}$, hinc $q\,dx = dp = \frac{q\,dq}{Q}$, ita vt sit $x = \int \frac{dq}{Q}$ et $p = \int \frac{q\,dq}{Q}$;

vnde colligimus $pdx=dy=\frac{dq}{Q}\int\frac{qdq}{Q}$, ergo $y=\int\frac{dq}{Q}\int\frac{qdq}{Q}$. Quare per eandem variabilem q vtraque variabilis x et y ita determinatur, vt fit

$$x=\int\frac{dq}{Q} \text{ et } y=\int\frac{dq}{Q}\int\frac{qdq}{Q}.$$

Aequatio quarta $s=R=\frac{dr}{dx}$ dat $dx=\frac{dr}{R}$, vnde colligimus $rdx=dq=\frac{rdr}{R}$ ita vt fit $q=\int\frac{rdr}{R}$. Porro $qdx=dp$ dat $dp=\frac{dr}{R}\int\frac{rdr}{R}$, hincque $p=\int\frac{dr}{R}\int\frac{rdr}{R}$; et quia $pdx=dy$ habebimus $dy=\frac{dr}{R}\int\frac{dr}{R}\int\frac{rdr}{R}$, quare per r ambae variabiles principales x et y ita definiuntur:

$$x=\int\frac{dr}{R} \text{ et } y=\int\frac{dr}{R}\int\frac{dr}{R}\int\frac{rdr}{R}.$$

Aequatio quinta $t=S$ fimili modo tractata praebet:

$$x=\int\frac{ds}{S} \text{ et } y=\int\frac{ds}{S}\int\frac{ds}{S}\int\frac{ds}{S}\int\frac{sds}{S}$$

ficque facile vlterius progredi licet.

Coroll. 1.

1111. Ex formula fecunda intelligitur, fi x aequetur functioni ipfius p, vt fit $x=P$, fore $y=\int pdP=Pp-\int Pdp$, quod quidem per fe eft manifeftum.

Coroll. 2.

1112. Sin autem fit $x=Q$ ob $dx=dQ$ erit $qdx=dp=qdQ$ et $p=\int qdQ$, hincque $y=\int dQ\int qdQ$, feu $y=Q\int qdQ-\int qQdQ$. Vel etiam cum fit $y=\int dQ(qQ-\int Qdq)$ erit $y=\frac{1}{2}qQQ+\frac{1}{2}\int QQdq-Q\int Qdq$ fiue

fiue hoc modo

$$2y = QQq - 2Q\int Qdq + \int QQdq.$$

Coroll. 3.

1113. Simili modo fi $x = R$; erit

$q = \int rdx = \int rdR$ et $p = \int qdx = \int dR \int rdR$, atque

$y = \int pdx = \int dR \int dR \int rdR$, feu

$2y = \int dR(RRr - 2R\int Rdr + \int RRdr)$

per praeced. Coroll. Ergo per fimiles reductiones

$$6y = R^3 r - 3R^2 \int Rdr + 3R\int RRdr - \int R^3 dr.$$

Coroll. 4.

1114. At fi fuerit $x = S$ reperietur per fimiles reductiones:

$$24y = S^4 s - 4S^3 \int Sds + 6S^2 \int SSds - 4S\int S^3 ds + \int S^4 ds$$

hinc ergo per differentiationes retrogrediendo:

$$24pdS = 4S^3 s dS - 12SSdS\int Sds + 12SdS\int SSds - 4dS\int S^3 ds$$

feu $6p = S^3 s - 3SS\int Sds + 3S\int SSds - \int S^3 ds$ et

$$2q = S^2 s - 2S\int Sds + \int SSds$$

tum $r = Ss - \int Sds$ et $s = s$.

Problema 143.

1115. Iisdem manentibus denominationibus, quibus hactenus fumus vfi, inueftigare integralia

 harum

harum formularum simpliciorum:

$$q=Y;\ r=P;\ s=Q;\ t=R \text{ etc.}$$

Solutio.

Pro formula prima $q=Y$; cum sit $q=\frac{p\,dp}{dy}$ erit $p\,dp=Y\,dy$ et $pp=2\int Y\,dy$ hinc $p=\sqrt{2\int Y\,dy}=\frac{dy}{dx}$, vnde colligitur $x=\int\frac{dy}{\sqrt{2\int Y\,dy}}$, sicque x per y determinatur.

Pro formula secunda $r=P$ ob $r=\frac{q\,dq}{dp}$, habebimus $q\,dq=P\,dp$ et $q=\sqrt{2\int P\,dp}=\frac{p\,dp}{dy}=\frac{dp}{dx}$, vnde concludimus

$$x=\int\frac{dp}{\sqrt{2\int P\,dp}} \text{ et } y=\int\frac{p\,dp}{\sqrt{2\int P\,dp}}.$$

Pro formula tertia $s=Q$ ob $s=\frac{r\,dr}{dq}$, fiet $r=\sqrt{2\int Q\,dq}=\frac{q\,dq}{dp}$, vnde sequitur $p=\int\frac{q\,dq}{\sqrt{2\int Q\,dq}}$. Cum vero sit $r=\frac{dq}{dx}$, erit $dx=\frac{dq}{\sqrt{2\int Q\,dq}}$, et ob $p\,dx=dy$ habebimus:

$$x=\int\frac{dq}{\sqrt{2\int Q\,dq}} \text{ et } y=\int\frac{dq}{\sqrt{2\int Q\,dq}}\int\frac{q\,dq}{\sqrt{2\int Q\,dq}}.$$

Pro formula quarta $t=R$ ob $t=\frac{s\,ds}{dr}$, nanciscimur $s=\sqrt{2\int R\,dr}$. At est $s=\frac{dr}{dx}$, vnde fit $dx=\frac{dr}{\sqrt{2\int R\,dr}}$. Est vero etiam $s=\frac{r\,dr}{dq}$, ideoque $q=\int\frac{r\,dr}{\sqrt{2\int R\,dr}}$ sed quoniam $p=\int q\,dx$, fit $p=\int\frac{dr}{\sqrt{2\int R\,dr}}\int\frac{r\,dr}{\sqrt{2\int R\,dr}}$, ex quo prodit $y=\int p\,dx$. Quocirca x et y ita per r determinantur vt sit:

$$x=\int\frac{dr}{\sqrt{2\int R\,dr}} \text{ et } y=\int\frac{dr}{\sqrt{2\int R\,dr}}\int\frac{dr}{\sqrt{2\int R\,dr}}\int\frac{r\,dr}{\sqrt{2\int R\,dr}}.$$

Pro

Pro formula quinta $u=S$, ob $u=\frac{t\,dt}{ds}$ adipiscimur $t=\sqrt{2\int S\,ds}=\frac{ds}{dx}$, vt fit $dx=\frac{ds}{\sqrt{2\int S\,ds}}$. Est vero etiam $t=\frac{s\,ds}{dr}$, ergo $r=\int\frac{s\,ds}{\sqrt{2\int S\,ds}}$. Tum $q=\int r\,dx$, $p=\int q\,dx$ et $y=\int p\,dx$, ex quibus conficitur:

$$x=\int\frac{ds}{\sqrt{2\int S\,ds}} \text{ et } y=\int\frac{ds}{\sqrt{2\int S\,ds}}\int\frac{ds}{\sqrt{2\int S\,ds}}\int\frac{ds}{\sqrt{2\int S\,ds}}\int\frac{s\,ds}{\sqrt{2\int S\,ds}}.$$

Scholion.

1116. Hi sunt casus, quibus formulas illas simpliciores, supra recensitas resoluere licet, neque methodus patet, qua reliquae tractari queant. Multo pauciores occurrunt casus tractabiles in formis magis compositis, vbi $\frac{d^n y}{dx^n}$ aequatur functioni binarum pluriumue quantitatum variabilium, ob quam penuriam parum admodum suppetit, quod in hac sectione exponere queamus. Aequationum autem, quae per methodos adhuc inuentas tractari possunt, haec fere est forma generalis

$$Ay+B.\frac{dy}{dx}+C.\frac{ddy}{dx^2}+D.\frac{d^3y}{dx^3}+\text{etc.}=0$$

sumto elemento dx constante, quae etiam ponendo

$$dy=p\,dx,\ dp=q\,dx,\ dq=r\,dx \text{ etc.}$$

ita repraesentari potest:

$$Ay+Bp+Cq+Dr+Es+\text{etc.}=0.$$

 Deinde

Deinde vero etiam aequationes hac forma latius patente contentae resolutionem admittunt:

$$Ay + Bp + Cq + Dr + Es + \text{etc.} = X$$

denotante X functionem quamcunque ipsius x. Porro quoque sequentes formae, quae quidem ad illas reduci possunt, ad integrationem perduci possunt:

$$Ay + \frac{Bp}{x} + \frac{Cq}{x^2} + \frac{Dr}{x^3} + \frac{Es}{x^4} \text{ etc.} = 0 \text{ et}$$

$$Ay + \frac{Bp}{x} + \frac{Cq}{xx} + \frac{Dr}{x^3} + \frac{Es}{x^4} + \text{etc.} = X$$

quarum resolutio adeo succedit, ad quantumuis gradum etiam differentialitas assurgat; in harum ergo aequationum euolutione tractatio nostra versabitur.

CAPVT II.

CAPVT II.

DE RESOLVTIONE AEQVATIONVM HVIVS FORMAE

$$Ay + B.\frac{dy}{dx} + C.\frac{ddy}{dx^2} + D.\frac{d^3y}{dx^3} + E.\frac{d^4y}{dx^4} + \text{etc.} = 0$$

SVMTO ELEMENTO dx CONSTANTE.

Problema 144.

1117.

Aequationis differentialis tertii gradus:

$$Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} = 0$$

ſumto elemento dx conſtante, integrale completum inuenire.

Solutio.

Cum A, B, C, D ſint quantitates conſtantes, leui attentione adhibita, patet, iſti aequationi huiusmodi formam $y = e^{\lambda x}$ ſatisfacere; cum enim hinc fit

$$\frac{dy}{dx} = \lambda e^{\lambda x};\ \frac{ddy}{dx^2} = \lambda^2 e^{\lambda x};\ \frac{d^3y}{dx^3} = \lambda^3 e^{\lambda x}$$

his ſubſtitutis et aequatione per $e^{\lambda x}$ diuiſa fit:

$$A + \lambda B + \lambda^2 C + \lambda^3 D = 0$$

vnde

vnde exponens λ determinatur, qui cum tres valores sortiatur, qui sint α, β, γ habebimus tres formulas satisfacientes $y = e^{\alpha x}$; $y = e^{\beta x}$; $y = e^{\gamma x}$. Verum ex natura aequationis propositae perspicuum est, si ei satisfaciant valores $y = P$, $y = Q$, $y = R$, etiam his vtcunque coniungendis satisfacturum

$$y = \mathfrak{A}P + \mathfrak{B}Q + \mathfrak{C}R.$$

Quare ex ternis formulis inuentis nanciscimur hanc latissime patentem aeque satisfacientem

$$y = \mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x}$$

quae forma cum tres constantes arbitrarias $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ complectatur, reuera erit integrale completum aequationis nostrae propositae.

Coroll. 1.

1118. Integrale ergo completum tot constat partibus, quot radices habeat seu factores aequatio

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0$$

cuius si factor fuerit $a + \lambda$, pars integralis erit e^{-ax}.

Coroll. 2.

1119. Haec scilicet pars erit integrale huius aequationis $ay + \frac{dy}{dx} = 0$. Vnde si sit

$$A + B\lambda + C\lambda^2 + D\lambda^3 = (a+\lambda)(b+\lambda)(c+\lambda)$$

quae-

quaerantur valores ipsius y ex his aequationibus simplicibus

$$ay+\frac{dy}{dx}=0\,,\quad by+\frac{dy}{dx}=0\,,\quad cy+\frac{dy}{dx}=0\,,$$

qui si sint $y=P$, $y=Q$, $y=R$, integrale aequationis propositae erit

$$y=\mathfrak{A}P+\mathfrak{B}Q+\mathfrak{C}R.$$

Coroll. 3.

1120. Si binae radices sint aequales puta $\beta=\alpha$, consideretur differentia vt euanescens $\beta=\alpha+\omega$, et cum sit

$$e^{\beta x}=e^{\alpha x}.e^{\omega x}=e^{\alpha x}(1+\omega x)\,,$$

euidens est loco $\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\beta x}$ scribi debere

$$\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\alpha x}x=e^{\alpha x}(\mathfrak{A}+\mathfrak{B}x)\,,$$

ac si omnes tres radices fuerint aequales $\alpha=\beta=\gamma$, vt aequatio sit:

$$y+\frac{3\,dy}{a\,dx}+\frac{3\,ddy}{a^2\,dx^2}+\frac{d^3y}{a^3\,dx^3}=0\,,$$

ob $\alpha=\beta=\gamma=-a$, integrale completum erit

$$e^{-ax}(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}xx).$$

Coroll. 4.

1121. Si binae radices fiant imaginariae, vt sit $\alpha=\mu+\nu\sqrt{-1}$ et $\beta=\mu-\nu\sqrt{-1}$, loco $\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\beta x}$ scribi debet

$$e^{\mu x}(\mathfrak{A}e^{\nu x\sqrt{-1}}+\mathfrak{B}e^{-\nu x\sqrt{-1}})\,,$$

 quae

quae reducitur ad hanc formam

$$e^{\mu x}(\mathfrak{A}\operatorname{cof.}\nu x + \mathfrak{B}\operatorname{fin.}\nu x).$$

Scholion 1.

1122. Quanquam aequatio propofita triplicem integrationem poftulat, antequam ad relationem finitam inter x et y perueniatur, hic tamen vna operatione, quae ne integrationi quidem eft affinis, eo pertigimus. Coniectura fcilicet formam collegimus aequationi particulariter fatisfacientem, fimulque tres huiusmodi formas fumus confecuti. Deinde ex ipfa aequationis indole intelleximus, fi valores finguli $y = P$, $y = Q$, $y = R$ fatisfaciant, etiam formam ex iis compofitam $y = \mathfrak{A}P + \mathfrak{B}Q + \mathfrak{C}R$ fatisfacere debere, quod nifi commode eueniffet, ex illis ternis valoribus nihil amplius concludi potuiffet. Ex eodem ergo principio in genere huiusmodi aequationes differentiales, quoticunque fuerint gradus, vno quafi actu ita refolui poterunt, vt adeo integrale completum affignetur.

Scholion 2.

1123. Quoniam aequationem differentialem tertii gradus

$$Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} = 0$$

in genere refoluere licuit, vt integrale completum effet

$$y = \mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x} + \mathfrak{C}e^{\gamma x},$$

exiften-

existentibus α, β, γ radicibus huius aequationis cubicae

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0,$$

hinc vsum non contemnendum pro aliis aequationibus, in quas illam transformare licet, percipiemus. Primo autem illam aequationem ad differentialem secundi gradus reuocare licet ope substitutionis $y = e^{\int u dx}$; vnde fit

$$\frac{dy}{dx} = e^{\int u dx} u; \quad \frac{ddy}{dx^2} = e^{\int u dx}\left(\frac{du}{dx} + uu\right) \text{ et}$$
$$\frac{d^3 y}{dx^3} = e^{\int u dx}\left(\frac{ddu}{dx^2} + \frac{3 u du}{dx} + u^3\right),$$

ita vt aequatio transformata sit diuisione per $e^{\int u dx}$ facta:

$$A + Bu + Cuu + Du^3 + C\frac{du}{dx} + \frac{3 D u du}{dx} + \frac{D ddu}{dx^2} = 0$$

cuius ergo ob $u = \frac{dy}{y dx}$ integrale est

$$u = \frac{\alpha \mathfrak{A} e^{\alpha x} + \beta \mathfrak{B} e^{\beta x} + \gamma \mathfrak{C} e^{\gamma x}}{\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x}}.$$

Haec autem aequatio vlterius ad gradum primum reducitur ponendo $dx = \frac{du}{t}$, cum enim elementum dx sumtum sit constans, erit $tddu - dtdu = 0$ seu $ddu = \frac{dtdu}{t}$, vnde fit $\frac{du}{dx} = t$ et $\frac{ddu}{dx^2} = \frac{tdt}{du}$, ita vt prodeat haec aequatio differentialis primi gradus:

$$A + Bu + Cuu + Du^3 + t(C + 3Du) + \frac{D t dt}{du} = 0$$

cuius ergo resolutio quoque est in potestate: vtriusque scilicet variabilis u et t valor per candem va-

 ria-

riabilem x exprimi poteſt. Cum enim y per x detur, erit primo $u = \frac{dy}{y\,dx}$ tum vero $t + uu = \frac{ddy}{y\,dx^2}$: ob $\frac{du}{dx} = t$. Loco y ergo ſubſtituto valore ſupra invento erit

$$u = \frac{\alpha \mathfrak{A} e^{\alpha x} + \beta \mathfrak{B} e^{\beta x} + \gamma \mathfrak{C} e^{\gamma x}}{\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x}} \text{ et}$$

$$t + uu = \frac{\alpha\alpha \mathfrak{A} e^{\alpha x} + \beta\beta \mathfrak{B} e^{\beta x} + \gamma\gamma \mathfrak{C} e^{\gamma x}}{\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x}}$$

dummodo α, β, γ ſint radices ex hac aequatione

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0.$$

Obſeruari autem conuenit illam aequationem poſito $t + uu = z$, abire in hanc formam

$$A + Bu + z(C + Du) + \frac{D\,dz}{du}(z - uu) = 0$$

quae latius patere videtur, quam illae eiusdem generis aequationes, quas ſupra tractauimus; cuius, quia ratio per methodos cognitas integrandi non conſtat, reſolutio facillime inſtituitur ponendo

$$u = \frac{dy}{y\,dx} \text{ et } z = \frac{ddy}{y\,dx^2}$$

vnde fit

$$du = \frac{ddy}{y\,dx} - \frac{dy^2}{yy\,dx} \text{ et } dz = \frac{d^3y}{y\,dx^2} - \frac{dy\,ddy}{yy\,dx^2}$$

ideoque

$$\frac{dz}{du} = \frac{y\,d^3y - dy\,ddy}{dx(y\,ddy - dy^2)} \text{ et } z - uu = \frac{y\,ddy - dy^2}{yy\,dx^2}$$

ſicque

ficque refultat haec aequatio

$$A+\frac{Bdy}{ydx}+\frac{Cddy}{ydx^2}+\frac{Ddyddy}{yydx^3}+\frac{Dyd^3y-Ddyddy}{yydx^3}=0 \text{ feu}$$

$$Ay+B\frac{dy}{dx}+C\frac{ddy}{dx^2}+D\frac{d^3y}{dx^3}=0$$

cuius refolutio eft oftenfa.

Scholion 3.

1124. Aequatio illa differentialis primi gradus

$$Dtdt+tdu(C+3Du)+du(A+Bu+Cu^2+Du^3)=0$$

cuius integrale inuenimus, diligentiore euolutione eft digna. Ac primo quidem obferuo eam integrabilem reddi, fi diuidatur per hanc formam:

$$DDt^2+Dtt(B+2Cu+3Duu)+t(C+3Du)(A+Bu+Cu^2+Du^3)$$
$$+(A+Bu+Cu^2+Du^3)^2$$

vnde concludimus et hanc aequationem

$$Dzdz-Duudz+zdu(C+Du)+du(A+Bu)=0$$

integrabilem fieri, fi diuidatur per hanc formam

$$DDz^2+Dzz(B+2Cu)+z(AC+(3AD+BC)u+(BD+CCuu))$$
$$+AA+2ABu+(AC+BB)uu+(AD+BC)u^3.$$

Vtrinque autem diuifor ifte nihilo aequatus praebet integrale particulare, vnde cum t vel. z ternos obtineant valores, finguli exhibebunt integralia particularia. Hinc operae pretium erit in genere aequationem

$$ydy+yPdx+Qdx=0$$

inueſtigare quae per formam

$$y^3 + Lyy + My + N$$

diuiſa integrabilis euadat. Per operationem autem ſupra explicatam inuenitur:

$$dL = 2Pdx;\ dM = PLdx + 3Qdx;\ dN = 2QLdx \text{ et } PN - QM = 0;$$

vnde colligitur:

$$Pdx = \tfrac{1}{2}dL;\ Qdx = \tfrac{dN}{2L};\ dM = \tfrac{1}{2}LdL + \tfrac{3dN}{2L} \text{ et } NdL = \tfrac{MdN}{L} \text{ ſeu } M = \tfrac{NLdL}{dN};$$

qui valor ibi ſubſtitutus ſumto dN conſtante prodit

$$3dN^2 = LLdLdN + 2NLLddL + 2NLdL^2$$

quae per dL multiplicata, tranſit in

$$3dLdN^2 = d.NL^2dL^2.$$

Verum commodius, ac ſingulari quidem modo illae aequationes reſoluuntur, ſtatuendo

$$N = \alpha Z^2 \text{ et } L = \tfrac{dZ}{dz};$$

vnde ſumto elemento dz conſtante deducitur $M = \frac{ZddZ}{2\alpha dz^2}$; hincque

$$dM = \tfrac{Zd^3Z + dZddZ}{2dz^2} \text{ et } \tfrac{1}{2}LdL + \tfrac{3dN}{2L} = \tfrac{dZddZ}{2dz^2} + 3\alpha Zdz.$$

Ergo $d^3Z = 6\alpha dz^3$, ideoque

$$Z = \alpha z^3 + \beta zz + \gamma z + \delta;\ Pdx = \tfrac{ddZ}{2dz} \text{ et } Qdx = \alpha Zdz.$$

Quocirca ſumto $Z = \alpha z^3 + \beta z^2 + \gamma z + \delta$ haec aequatio

$$ydy + y.\tfrac{ddZ}{2dz} + \alpha Zdz = 0$$

inte-

integrabilis redditur diuisa per hanc formam:

$$y^3 + y^2 . \frac{dZ}{dz} + y . \frac{Z ddZ}{2dz^2} + \alpha ZZ.$$

Praeterea si Z habeat factores vt proposita sit haec aequatio:

$$ydy + ydz(\alpha + \beta + \gamma + 3z) + dz(\alpha + z)(\beta + z)(\gamma + z) = 0$$

diuisor eam integrabilem reddens erit

$$(y + (\alpha + z)(\beta + z))(y + (\alpha + z)(\gamma + z))(y + (\beta + z)(\gamma + z))$$

cuius singuli factores nihilo aequati praebent integrale particulare. Ex vnoquoque autem, more magis consueto integrale completum ita elicitur. Ponatur

$$y = v - (\alpha + z)(\beta + z),$$

ac reperitur:

$$vdv + vdz(\gamma + z) - dv(\alpha + z)(\beta + z) = 0$$

sit porro $dv = pdz$ eritque $v = \frac{p(\alpha + z)(\beta + z)}{p + \gamma + z}$, et differentiando locoque dv ponendo pdz, orietur haec aequatio

$$dp(\alpha + z)(\beta + z)(\gamma + z) = dz(p^3 + (2\gamma - \alpha - \beta)p^2 + (\gamma - \alpha)(\gamma - \beta)p)$$

quae dat hanc separatam

$$\frac{dz}{(\alpha + z)(\beta + z)(\gamma + z)} = \frac{dp}{p(p + \gamma - \alpha)(p + \gamma - \beta)}.$$

Problema 145.

1125. Aequationis differentialis cuiuscunque gradus

$$Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + E\frac{d^4y}{dx^4} + \text{etc.} = 0$$

sumto

sumto elemento dx constante, integrale completum inuenire.

Solutio.

Et huic aequationi euidens est satisfacere formulam $y = e^{\lambda x}$, cum enim hinc sit

$$\frac{dy}{dx} = \lambda e^{\lambda x};\ \frac{ddy}{dx^2} = \lambda^2 e^{\lambda x};\ \frac{d^3 y}{dx^3} = \lambda^3 e^{\lambda x}$$

et in genere $\frac{d^n y}{dx^n} = \lambda^n e^{\lambda x}$, facta substitutione peruenietur ad hanc aequationem, postquam scilicet per $e^{\lambda x}$ diuiserimus:

$$A + B\lambda + C\lambda^2 + D\lambda^3 + E\lambda^4 + \text{etc.} = 0$$

ex qua valorem ipsius λ definiri oportet. Hinc littera λ totidem valores obtinebit, quoti fuerit ordinis aequatio differentialis proposita, quorum singuli aequationi aeque satisfacient. Qui valores si sint α, β, γ, δ etc. integralia quidem particularia erunt

$$y = \mathfrak{A} e^{\alpha x};\ y = \mathfrak{B} e^{\beta x};\ y = \mathfrak{C} e^{\gamma x} \text{ etc.}$$

Verum ex ipsa aequationis natura perspicuum est aggregata quotcunque horum valorum, ideoque etiam omnium perinde satisfacere. Cum igitur aggregatum omnium:

$$y = \mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x} + \mathfrak{D} e^{\delta x} \text{ etc.}$$

tot complectatur constantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. quoti ordinis differentialis est aequatio proposita, quin haec forma eius sit integrale completum, dubitari nequit.

nequit. Afcendat aequatio differentialis ad gradum n vt fit

$$Ay+B\frac{dy}{dx}+C\frac{ddy}{dx^2}+D\frac{d^3y}{dx^3}+\ldots\ldots+N\frac{d^ny}{dx^n}=0$$

atque integrale completum ex n partibus conftabit, quas ex refolutione huius aequationis algebraicae ordinis n, fcilicet

$$A+B\lambda+C\lambda^2+D\lambda^3+\ldots\ldots+N\lambda^n=0$$

definiri oportet. Singuli nimirum eius factores fimplices partes illas patefacient, ita fi factor fit $\alpha-\lambda$ ex eo integralis pars nafcitur $\mathfrak{A}e^{\alpha x}$, quae vti manifeftum eft ex integratione aequationis differentialis fimplicis $\alpha y-\frac{dy}{dx}=0$ nafcitur. Simili modo duo factores coniunctim

$$(\alpha-\lambda)(\beta-\lambda)=\alpha\beta-(\alpha+\beta)\lambda+\lambda\lambda$$

integralis portionem $\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\beta x}$ fuppeditant, quae fimul eft integrale huius aequationis differentialis fecundi gradus

$$\alpha\beta y-(\alpha+\beta)\frac{dy}{dx}+\frac{ddy}{dx^2}=0.$$

Atque in genere fi aequationis illius algebraicae factor fit

$$a+b\lambda+c\lambda^2+f\lambda^3+\text{etc.}=0,$$

ex hoc viciffim formetur aequatio differentialis:

$$ay+b\frac{dy}{dx}+c\frac{ddy}{dx^2}+f\frac{d^3y}{dx^3}\text{ etc.}=0$$

cuius integrale completum fi fit $y=P$, id fimul

 erit

erit pars integralis aequationis propoſitae. Atque hoc modo ex ſingulis factoribus aequationis algebraicae

$$A + B\lambda + C\lambda^2 + D\lambda^3 + \ldots\ldots + N\lambda^n = 0$$

deriuabuntur ſingulae partes integralis quaeſiti, quae iunctae eius integrale completum conſtituent, ita vt praecipuum negotium reſolutioni huius aequationis innitatur.

Coroll. 1.

1126. Si igitur iſtius aequationis algebraicae omnes factores ſimplices fuerint reales ſimulque inaequales, integratio nullam habet difficultatem. Si enim factor ſimplex ſit $f + g\lambda$, integralis pars inde oriunda eſt $\mathfrak{A} e^{\frac{-fx}{g}}$.

Coroll. 2.

1127. Si bini factores ſimplices ſint aequales ſeu factor fuerit $(f + g\lambda)^2$, pars integralis inde oriunda eſt $e^{\frac{-fx}{g}}(\mathfrak{A} + \mathfrak{B}x)$. Si factor ſit cubus $(f + g\lambda)^3$, inde oritur pars integralis $e^{\frac{-fx}{g}}(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2)$, et ex factore biquadrato $(f + g\lambda)^4$ huiusmodi pars

$$e^{\frac{-fx}{g}}(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3).$$

et ita porro pro quotcunque factoribus aequalibus vti ex §. 1120. colligere licet.

Coroll. 3.

Coroll. 3.

1128. Si factores occurrant imaginarii, bini coniuncti exhibent factorem trinomium realem, cuius forma ita repraesentatur

$$ff + 2fg\lambda \operatorname{cos}.\zeta + gg\lambda\lambda,$$

vnde deducitur

$$\lambda = -\frac{f}{g}(\operatorname{cos}.\zeta \pm \sqrt{-1}.\operatorname{sin}.\zeta),$$

quo cum §. 1121. collato fit $\mu = \frac{-f\operatorname{cos}.\zeta}{g}$ et $\nu = \frac{f\operatorname{sin}.\zeta}{g}$. Ex quo pars integralis ex tali factore oriunda erit

$$e^{\frac{-fx\operatorname{cos}.\zeta}{g}}\left(\mathfrak{A}\operatorname{cos}.\frac{fx\operatorname{sin}.\zeta}{g} + \mathfrak{B}\operatorname{sin}.\frac{fx\operatorname{sin}.\zeta}{g}\right).$$

Coroll. 4.

1129. Si huiusmodi formae quadratum inter factores occurrat

$$(ff + 2fg\lambda\operatorname{cos}.\zeta + gg\lambda\lambda)^2,$$

seu duo huiusmodi factores sint aequales, considerentur quasi infinite parum discrepantes, vt in altero loco $\frac{f}{g}$ sit $\frac{f}{g}(1+\omega)$; et ob

$$e^{\frac{-fx\operatorname{cos}.\zeta}{g}(1+\omega)} = e^{\frac{-fx\operatorname{cos}.\zeta}{g}}\left(1 - \frac{\omega fx}{g}\operatorname{cos}.\zeta\right);$$

$$\operatorname{cos}.\frac{fx\operatorname{sin}.\zeta}{g}(1+\omega) = \operatorname{cos}.\frac{fx\operatorname{sin}.\zeta}{g} - \frac{\omega fx\operatorname{sin}.\zeta}{g}\operatorname{sin}.\frac{fx\operatorname{sin}.\zeta}{g} \text{ et}$$

$$\operatorname{sin}.\frac{fx\operatorname{sin}.\zeta}{g}(1+\omega) = \operatorname{sin}.\frac{fx\operatorname{sin}.\zeta}{g} + \frac{\omega fx\operatorname{sin}.\zeta}{g}\operatorname{cos}.\frac{fx\operatorname{sin}.\zeta}{g}:$$

ex hoc factore colligitur pars integralis

$$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{array}{l}\mathfrak{A}'\cos.\frac{fx\sin.\zeta}{g}-\mathfrak{A}'\frac{\omega fx\cos.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}-\mathfrak{A}'\frac{\omega fx\sin.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g.}\\+\mathfrak{B}'\sin.\frac{fx\sin.\zeta}{g}-\mathfrak{B}'\frac{\omega fx\cos.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g}+\mathfrak{B}'\frac{\omega fx\sin.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}\end{array}\right\}$$

cui prior addi debet. Hunc in finem constantes ita contrahamus ponendo

$$\mathfrak{A}+\mathfrak{A}'=\mathfrak{E}\,;\quad -\frac{\mathfrak{A}'\omega f\cos.\zeta}{g}+\frac{\mathfrak{B}'\omega f\sin.\zeta}{g}=\mathfrak{G}$$

$$\mathfrak{B}+\mathfrak{B}'=\mathfrak{F}\,;\quad -\frac{\mathfrak{A}'\omega f\sin.\zeta}{g}-\frac{\mathfrak{B}'\omega f\cos.\zeta}{g}=\mathfrak{H}$$

vnde illae constantes vtique determinantur, eritque pars integralis respondens:

$$e^{\frac{-fx\cos.\zeta}{g}}((\mathfrak{E}+\mathfrak{G}x)\cos.\frac{fx\sin.\zeta}{g}+(\mathfrak{F}+\mathfrak{H}x)\sin.\frac{fx\sin.\zeta}{g}.$$

Scholion.

1130. En ergo vniuersam methodum huiusmodi aequationum differentialium integralia inueniendi

$$Ay+B\frac{dy}{dx}+C\frac{ddy}{dx^2}+D\frac{d^3y}{dx^3}+\ldots\ldots+N\frac{d^ny}{dx^n}=0$$

ita in compendium contractam. Scribatur vt iste laterculus indicat:

loco	y	$\frac{dy}{dx}$	$\frac{ddy}{dx^2}$	$\frac{d^3y}{dx^3}$	$\frac{d^4y}{dx^4}$	.....	$\frac{d^ny}{dx^n}$
scribatur	1	z	z^2	z^3	z^4		z^n

vt oriatur haec aequatio algebraica

$$A+Bz+Cz^2+Dz^3+Ez^4+\ldots\ldots+Nz^n=0$$

cuius

cuius ſinguli factores reales, ſiue ſimplices ſiue duplicati notentur, atque inſuper caſus quibus duo pluresue ſunt inter ſe aequales probe obſeruentur. Tum cuiusmodi partes pro Integrali quaeſito ex ſingulis factoribus oriantur, ex ſequente Tabella intelligere licet:

Factores	Partes integralis
$f+gz$	$\mathfrak{A}e^{\frac{-fx}{g}}$
$(f+gz)^2$	$(\mathfrak{A}+\mathfrak{B}x)e^{\frac{-fx}{g}}$
$(f+gz)^3$	$(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2)e^{\frac{-fx}{g}}$
$(f+gz)^4$	$(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3)e^{\frac{-fx}{g}}$
$ff+2fgz\cos.\zeta+ggzz$	$e^{\frac{-fx\cos.\zeta}{g}}(\mathfrak{A}\cos.\frac{fx\sin.\zeta}{g}+\mathfrak{B}\sin.\frac{fx\sin.\zeta}{g})$
$(ff+2fgz\cos.\zeta+ggzz)^2$	$e^{\frac{-fx\cos.\zeta}{g}}\begin{cases}(\mathfrak{A}+\mathfrak{B}x)\cos.\frac{fx\sin.\zeta}{g}\\(\mathfrak{a}+\mathfrak{b}x)\sin\frac{fx\sin.\zeta}{g}\end{cases}$
$(ff+2fgz\cos.\zeta+ggzz)^3$	$e^{\frac{-fx\cos.\zeta}{g}}\begin{cases}\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}xx)\cos.\frac{fx\sin.\zeta}{g}\\(\mathfrak{a}+\mathfrak{b}x+\mathfrak{c}xx)\sin.\frac{fx\sin.\zeta}{g}\end{cases}$

etc.

Pro ſingulis autem factoribus diuerſae litterae conſtantes ſcribi debent, vt integrale omnibus numeris completum obtineatur.

Exemplum I.

1131. *Aequationis differentialis quarti gradus:*

$$y-\frac{2dy}{dx}+\frac{2ddy}{dx^2}-\frac{2d^3y}{dx^3}+\frac{d^4y}{dx^4}=0$$

integrale completum aſſignare.

Hinc oritur aequatio algebraica:

$$1-2z+2zz-2z^3+z^4=0$$

quae in hos factores resoluitur $(1-z)^2(1+zz)$, quorum prior ob $f=1$ et $g=-1$ praebet hanc partem integralis $(\mathfrak{A}+\mathfrak{B}x)e^x$, alter vero factor ob $f=1$; $\operatorname{cof.}\zeta=0$; $g=1$ et $\operatorname{fin.}\zeta=1$ dat: $\mathfrak{A}\operatorname{cof.}x+\mathfrak{B}\operatorname{fin.}x$. Quare integrale completum, quod quaeritur, erit

$$y=(\mathfrak{A}+\mathfrak{B}x)e^x+\mathfrak{C}\operatorname{cof.}x+\mathfrak{D}\operatorname{fin.}x$$

continens quatuor constantes arbitrarias. Quod si velimus vt posito $x=0$, fiat $y=0$, fieri oportet $\mathfrak{A}+\mathfrak{C}=0$, si etiam $\frac{dy}{dx}$ eodem casu euanescere debeat, ob

$$\frac{dy}{dx}=(\mathfrak{A}+\mathfrak{B}+\mathfrak{B}x)e^x-\mathfrak{C}\operatorname{fin.}x+\mathfrak{D}\operatorname{cof.}x$$

fieri debet $\mathfrak{A}+\mathfrak{B}+\mathfrak{D}=0$. Si praeterea $\frac{ddy}{dx^2}$ euanescere debeat, ob

$$\frac{ddy}{dx^2}=(\mathfrak{A}+2\mathfrak{B}+\mathfrak{B}x)e^x-\mathfrak{C}\operatorname{cof.}x-\mathfrak{D}\operatorname{fin.}x$$

fieri debet $\mathfrak{A}+2\mathfrak{B}-\mathfrak{C}=0$. Quare his tribus conditionibus satisfaciemus sumendo $\mathfrak{C}=-\mathfrak{A}$; $\mathfrak{B}=-\mathfrak{A}$ et $\mathfrak{D}=0$, ita vt sit integrale:

$$y=\mathfrak{A}(1-x)e^x-\mathfrak{A}\operatorname{cof.}x.$$

Exemplum 2.

1132. *Aequationem differentialem quarti ordinis*

$$Ay+C\frac{ddy}{dx^2}+E\frac{d^4y}{dx^4}=0$$

sumto elemento dx *constante integrare.*

Aequa-

Aequatio algebraica ad integrationem perducens eſt:

$$A + Czz + Ez^4 = 0$$

quae ſemper duos factores duplicatos reales habet, quorum forma duplex eſſe poteſt:

vel $(aa + 2maz + nzz)(aa - 2maz + nzz)$

vel $(aa + mzz)(aa + nzz)$.

Ex priori eſt:

$$A = a^4;\ C = 2naa - 4mmaa;\ E = nn$$

ex poſteriore vero

$$A = a^4;\ C = (m+n)aa;\ E = mn$$

ſemper autem terminum primum A biquadrato a^4 repraeſentare licet, et prior reſolutio locum habet, ſi E ſit numerus poſitiuus, et $2naa - C$ ſeu $2\sqrt{AE} - C$ quoque poſitiuus, ideoque $4AE > CC$. Poſterior vero ſi $CC < 4AE$. Tum igitur videndum eſt, ad quamnam claſſem ſinguli factores pertineant, vnde ſequentes caſus occurrent:

I. Si omnes quatuor factores ſimplices ſunt reales, erit

$$A + Czz + Ez^4 = (a+z)(a-z)(b+z)(b-z)$$

haec habebitur aequatio:

$$aabby - (aa+bb)\frac{ddy}{dx^2} + \frac{d^4y}{dx^4} = 0$$

cuius integrale completum eſt:

$$y = \mathfrak{A}e^{ax} + \mathfrak{B}e^{-ax} + \mathfrak{C}e^{bx} + \mathfrak{D}e^{-bx}.$$

Ac si sit $b=a$, huius aequationis

$$a^4y-\frac{2aaddy}{dx^2}+\frac{d^4y}{dx^4}=0$$

integrale completum erit

$$y=(\mathfrak{A}+\mathfrak{B}x)e^{ax}+(\mathfrak{C}+\mathfrak{D}x)e^{-ax}.$$

II Si duo factores simplices sint reales, duo vero imaginarii, erit

$$A+Czz+Ez^4=(a+z)(a-z)(bb+zz)$$

haec habebitur aequatio:

$$aabby+(aa-bb)\frac{ddy}{dx^2}-\frac{d^4y}{dx^4}=0$$

cuius integrale completum est:

$$y=\mathfrak{A}e^{ax}+\mathfrak{B}e^{-ax}+\mathfrak{C}\operatorname{cof.}bx+\mathfrak{D}\operatorname{fin.}bx.$$

III. Si omnes factores simplices sint imaginarii, duo casus sunt euoluendi:

1) si $A+Czz+Ez^4=(aa+zz)(bb+zz)$

vnde huius aequationis

$$aabby+(aa+bb)\frac{ddy}{dx^2}+\frac{d^4y}{dx^4}=0$$

integrale completum erit:

$$y=\mathfrak{A}\operatorname{cof.}ax+\mathfrak{B}\operatorname{fin.}ax+\mathfrak{C}\operatorname{cof.}bx+\mathfrak{D}\operatorname{fin.}bx$$

2) si $A+Czz+Ez^4=(aa+2az\operatorname{cof}\zeta+zz)(aa-2az\operatorname{cof.}\zeta+zz)$.

vnde huius aequationis

$$a^4y-\frac{2aaddy}{dx^2}\operatorname{cof.}2\zeta+\frac{d^4y}{dx^4}=0$$

inte-

integrale completum eſt

$$y = e^{+ax\,\text{coſ.}\,\zeta}(\mathfrak{A}\,\text{coſ.}\,a\,x\,\text{ſin.}\,\zeta + \mathfrak{B}\,\text{ſin.}\,a\,x\,\text{ſin.}\,\zeta)$$
$$+ e^{-ax\,\text{coſ.}\,\zeta}(\mathfrak{C}\,\text{coſ.}\,a\,x\,\text{ſin.}\,\zeta + \mathfrak{D}\,\text{ſin.}\,a\,x\,\text{ſin.}\,\zeta).$$

At ſi ſit priori caſu $b = a$ ſeu poſteriori $\text{coſ.}\,\zeta = 0$ huius aequationis

$$a^4 y + \frac{2aaddy}{dx^2} + \frac{d^4 y}{dx^4} = 0$$

integrale completum eſt

$$y = (\mathfrak{A} + \mathfrak{B}x)\,\text{coſ.}\,a\,x + (\mathfrak{C} + \mathfrak{D}x)\,\text{ſin.}\,a\,x$$

Scholion I.

1133. Cum igitur aequationis

$$Ay + \frac{Cddy}{dx^2} + \frac{Ed^4 y}{dx^4} = 0$$

integrale aſſignari poſſit, omnes aequationes quas inde deriuare licet, integrari poterunt. At haec aequatio per $2dy$ multiplicata primo per integrationem ad differentialem tertii ordinis reducitur:

$$Ayy + \frac{Cdy^2}{dx^2} + \frac{2Edyd^3y - Eddy^2}{dx^4} = \text{Conſt.}$$

In integrali autem ante inuento conſtantes $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ ita definire licet, vt haec conſtans Conſt. euaneſcat, ideoque huius aequationis

$$Ayy + \frac{Cdy^2}{dx^2} + \frac{2Edyd^3y - Eddy^2}{dx^4} = 0$$

integrale completum erit in noſtra poteſtate. Nunc ponatur $y = e^{\int v dx}$ vt ſit $v = \frac{dy}{ydx}$, et ob $\frac{dy}{dx} = e^{\int v dx} v$; $\frac{ddy}{dx^2} = e^{\int v dx}(\frac{dv}{dx} + vv)$ atque $\frac{d^3y}{dx^3} = e^{\int v dx}(\frac{ddv}{dx^2} + \frac{3vdv}{dx} + v^3)$,

aequatio nostra hanc induit formam:

$$A+Cvv+E\left(\frac{2vddv}{dx^2}+\frac{4vvdv}{dx}+v^4-\frac{dv^2}{dx^2}\right)=0.$$

Sit porro $dx=\frac{dv}{s}$, vt sit $s=\frac{dv}{dx}=\frac{ddy}{ydx^2}-\frac{dy^2}{yydx^2}$ erit $\frac{ddv}{dx}=ds$ et $\frac{ddv}{dx^2}=\frac{sds}{dv}$, vnde resultat haec aequatio differentialis primi gradus:

$$A+Cvv+E\left(\frac{2vsds}{dv}-ss+4vvs+v^4\right)=0$$

cuius relatio inter v et s ita ex relatione inter x et y inuenta definitur, vt sit

$$v=\frac{dy}{ydx} \text{ et } s=\frac{yddy-dy^2}{yydx^2}.$$

Scholion 2.

1134. Retenta autem illa constante per integrationem ingressa, vt habeatur:

$$Ayy+\frac{Cdy^2}{dx^2}+\frac{2Edyd^3y-Eddy^2}{dx^4}=G$$

in integrali completo, quo y per x exprimitur, constantes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ quantitati huic G conformiter determinari poterunt. Nunc igitur ponatur $dx=\frac{dy}{u}$, vt sit $\frac{dy}{dx}=u$, erit $\frac{ddy}{dx^2}=\frac{udu}{dy}$, et $\frac{d^3y}{dx^3}=d.\frac{udu}{dy}$ ideoque $\frac{d^3y}{dx^3}=\frac{u}{dy}d.\frac{udu}{dy}$. Vnde obtinetur haec aequatio differentialis secundi gradus:

$$Ayy+Cuu+E\left(\frac{2uu}{dy}d.\frac{udu}{dy}-\frac{uudu^2}{dy^2}\right)=G$$

vbi consideratio elementi pro constante assumti est exuta. Nihil ergo impedit quo minus sumamus dy pro

pro constante, fietque

$$Ayy + Cuu + E\left(\frac{2u^3 ddu}{dy^2} + \frac{uudu^2}{dy^2}\right) = G$$

quae ergo aequatio etiam integrari potest.
Vel si ponamus $yy = p$ et $uu = q$, sumto elemento dp constante, prodibit haec aequatio:

$$Ap + Cq + E\left(\frac{4pqddq + 2qdpdq - pdq^2}{dp^2}\right) = G.$$

Vel si in illa aequatione ponatur $u = r^{\frac{2}{3}}$, erit

$$Ayy + Cr^{\frac{4}{3}} + \tfrac{4}{3}Er^{\frac{5}{3}}.\frac{ddr}{dy^2} = G.$$

Quarum formarum integratio sine hoc subsidio maxime ardua videtur.

Problema 146.

1135. Proposita aequatione differentiali ordinis cuiuscunque $a^n y \pm \frac{d^n y}{dx^n} = 0$, vbi elementum dx constans est assumtum, eius integrale completum inuestigare.

Solutio.

Aequatio algebraica solutioni inseruiens, est $a^n \pm z^n = 0$ pro cuius resolutione duos casus considerari conuenit, prout signum vel superius vel inferius valeat.

I. Valeat superius, vt haec proposita sit aequatio

$$a^n y + \frac{d^n y}{dx^n} = 0$$

 et

et formulae $a^n + z^n$ factores reales sunt

$$aa - 2az\,\mathrm{cos}.\tfrac{\pi}{n} + zz;\quad aa - 2az\,\mathrm{cos}.\tfrac{3\pi}{n} + zz;\quad aa - 2az\,\mathrm{cos}.\tfrac{5\pi}{n} + zz;\ \text{etc.}$$

quorum vltimus est vel $aa - 2az\,\mathrm{cos}.\tfrac{n\pi}{n} + zz$ vel $aa - 2az\,\mathrm{cos}.\tfrac{(n-1)\pi}{n} + zz$, prout vel n vel $n-1$ fuerit numerus impar, atque illo quidem casu loco factoris quadrati $aa + 2az + zz$ eius radix $a + z$ sumi debet.

Hinc istius aequationis integrale completum est

$$y = +e^{ax\,\mathrm{cos}.\frac{\pi}{n}}\left(\mathfrak{A}\,\mathrm{cos}.\,ax\,\mathrm{sin}.\tfrac{\pi}{n} + \mathfrak{B}\,\mathrm{sin}.\,ax\,\mathrm{sin}.\tfrac{\pi}{n}\right)$$
$$+e^{ax\,\mathrm{cos}.\frac{3\pi}{n}}\left(\mathfrak{C}\,\mathrm{cos}.\,ax\,\mathrm{sin}.\tfrac{3\pi}{n} + \mathfrak{D}\,\mathrm{sin}.\,ax\,\mathrm{sin}.\tfrac{3\pi}{n}\right)$$
$$+e^{ax\,\mathrm{cos}.\frac{5\pi}{n}}\left(\mathfrak{E}\,\mathrm{cos}.\,ax\,\mathrm{sin}.\tfrac{5\pi}{n} + \mathfrak{F}\,\mathrm{sin}.\,ax\,\mathrm{sin}.\tfrac{5\pi}{n}\right)$$

etc.

cuius expressionis, si n sit numerus impar, vltima pars sit $\mathfrak{N}e^{-ax}$. Quod integrale etiam ita potest exhiberi

$$y = \mathfrak{A}e^{ax\,\mathrm{cos}.\frac{\pi}{n}}\,\mathrm{cos}.\left(ax\,\mathrm{sin}.\tfrac{\pi}{n} + \mathfrak{a}\right) + \mathfrak{B}e^{ax\,\mathrm{cos}.\frac{3\pi}{n}}\,\mathrm{cos}.\left(ax\,\mathrm{sin}.\tfrac{3\pi}{n} + \mathfrak{b}\right)$$
$$+\mathfrak{C}e^{ax\,\mathrm{cos}.\frac{5\pi}{n}}\,\mathrm{cos}.\left(ax\,\mathrm{sin}.\tfrac{5\pi}{n} + \mathfrak{c}\right) + \mathfrak{D}e^{ax\,\mathrm{cos}.\frac{7\pi}{n}}\,\mathrm{cos}.\left(ax\,\mathrm{sin}.\tfrac{7\pi}{n} + \mathfrak{d}\right)$$

etc.

quae forma eousque continuari debet, quoad similes termini recurrant.

II.

II. Si valeat fignum inferius, propofitaque fit haec aequatio:

$$a^n y - \frac{d^n y}{dx^n} = 0$$

formulae $a^n - z^n$ factores reales funt:

$$a - z;\ aa - 2az\cos.\tfrac{2\pi}{n} + zz;\ aa - 2az\cos.\tfrac{4\pi}{n} + zz;$$
$$aa - 2az\cos.\tfrac{6\pi}{n} + zz;\ \text{etc.}$$

quorum fi n numerus par, vltimus eft $a + z$, fin autem impar

$$aa - 2az\cos.\tfrac{(n-1)\pi}{n} + zz.$$

Quare aequationis huius integrale completum eft

$$y = \mathfrak{A}e^{ax} + e^{ax\cos.\frac{2\pi}{n}}\left(\mathfrak{B}\cos.ax\sin.\tfrac{2\pi}{n} + \mathfrak{C}\sin.ax\sin.\tfrac{2\pi}{n}\right)$$
$$+ e^{ax\cos.\frac{4\pi}{n}}\left(\mathfrak{D}\cos.ax\sin.\tfrac{4\pi}{n} + \mathfrak{E}\sin.ax\sin.\tfrac{4\pi}{n}\right)$$
$$+ e^{ax\cos.\frac{6\pi}{n}}\left(\mathfrak{F}\cos.ax\sin.\tfrac{6\pi}{n} + \mathfrak{G}\sin.ax\sin.\tfrac{6\pi}{n}\right)$$
$$\text{etc.}$$

quod integrale etiam ita exprimi poteft:

$$y = \mathfrak{A}e^{ax} + \mathfrak{B}e^{ax\cos.\frac{2\pi}{n}}\cos.\left(ax\sin.\tfrac{2\pi}{n} + \mathfrak{b}\right)$$
$$+ \mathfrak{C}e^{ax\cos.\frac{4\pi}{n}}\cos.\left(ax\sin.\tfrac{4\pi}{n} + \mathfrak{c}\right) + \mathfrak{D}e^{ax\cos.\frac{6\pi}{n}}\cos.\left(ax\sin.\tfrac{6\pi}{n} + \mathfrak{d}\right)$$
$$\text{etc.}$$

quae forma eousque eft continuanda, quamdiu termini a prioribus diuerfi prodeunt.

 Scho-

sumto elemento dx constante, integrale completum inuenire.

Solutio.

Et huic aequationi euidens est satisfacere formulam $y = e^{\lambda x}$, cum enim hinc sit

$$\frac{dy}{dx} = \lambda e^{\lambda x};\ \frac{ddy}{dx^2} = \lambda^2 e^{\lambda x};\ \frac{d^3y}{dx^3} = \lambda^3 e^{\lambda x}$$

et in genere $\frac{d^n y}{dx^n} = \lambda^n e^{\lambda x}$, facta substitutione peruenietur ad hanc aequationem, postquam scilicet per $e^{\lambda x}$ diuiserimus:

$$A + B\lambda + C\lambda^2 + D\lambda^3 + E\lambda^4 + \text{etc.} = 0$$

ex qua valorem ipsius λ definiri oportet. Hinc littera λ totidem valores obtinebit, quoti fuerit ordinis aequatio differentialis proposita, quorum singuli aequationi aeque satisfacient. Qui valores si sint α, β, γ, δ etc. integralia quidem particularia erunt

$$y = \mathfrak{A} e^{\alpha x};\ y = \mathfrak{B} e^{\beta x};\ y = \mathfrak{C} e^{\gamma x} \text{ etc.}$$

Verum ex ipsa aequationis natura perspicuum est aggregata quotcunque horum valorum, ideoque etiam omnium perinde satisfacere. Cum igitur aggregatum omnium:

$$y = \mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x} + \mathfrak{D} e^{\delta x} \text{ etc.}$$

tot complectatur constantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. quoti ordinis differentialis est aequatio proposita, quin haec forma eius sit integrale completum, dubitari nequit.

nequit. Afcendat aequatio differentialis ad gradum n vt fit

$$Ay+B\frac{dy}{dx}+C\frac{ddy}{dx^2}+D\frac{d^3y}{dx^3}+\ldots\ldots+N\frac{d^ny}{dx^n}=0$$

atque integrale completum ex n partibus conftabit, quas ex refolutione huius aequationis algebraicae ordinis n, fcilicet

$$A+B\lambda+C\lambda^2+D\lambda^3+\ldots\ldots+N\lambda^n=0$$

definiri oportet. Singuli nimirum eius factores fimplices partes illas patefacient, ita fi factor fit $\alpha-\lambda$ ex eo integralis pars nafcitur $\mathfrak{A}e^{\alpha x}$, quae vti manifeftum eft ex integratione aequationis differentialis fimplicis $\alpha y-\frac{dy}{dx}=0$ nafcitur. Simili modo duo factores coniunctim

$$(\alpha-\lambda)(\beta-\lambda)=\alpha\beta-(\alpha+\beta)\lambda+\lambda\lambda$$

integralis portionem $\mathfrak{A}e^{\alpha x}+\mathfrak{B}e^{\beta x}$ fuppeditant, quae fimul eft integrale huius aequationis differentialis fecundi gradus

$$\alpha\beta y-(\alpha+\beta)\frac{dy}{dx}+\frac{ddy}{dx^2}=0.$$

Atque in genere fi aequationis illius algebraicae factor fit

$$a+b\lambda+c\lambda^2+f\lambda^3+\text{etc.}=0,$$

ex hoc viciffim formetur aequatio differentialis:

$$ay+b\frac{dy}{dx}+c\frac{ddy}{dx^2}+f\frac{d^3y}{dx^3}\text{ etc.}=0$$

cuius integrale completum fi fit $y=P$, id fimul erit

erit pars integralis aequationis propositae. Atque hoc modo ex singulis factoribus aequationis algebraicae

$$A + B\lambda + C\lambda^2 + D\lambda^3 + \ldots\ldots + N\lambda^n = 0$$

deriuabuntur singulae partes integralis quaesiti, quae iunctae eius integrale completum constituent, ita vt praecipuum negotium resolutioni huius aequationis innitatur.

Coroll. 1.

1126. Si igitur istius aequationis algebraicae omnes factores simplices fuerint reales simulque inaequales, integratio nullam habet difficultatem. Si enim factor simplex sit $f + g\lambda$, integralis pars inde oriunda est $\mathfrak{A}e^{\frac{-fx}{g}}$.

Coroll. 2.

1127. Si bini factores simplices sint aequales seu factor fuerit $(f + g\lambda)^2$, pars integralis inde oriunda est $e^{\frac{-fx}{g}}(\mathfrak{A} + \mathfrak{B}x)$. Si factor sit cubus $(f + g\lambda)^3$, inde oritur pars integralis $e^{\frac{-fx}{g}}(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2)$, et ex factore biquadrato $(f + g\lambda)^4$ huiusmodi pars

$$e^{\frac{-fx}{g}}(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3).$$

et ita porro pro quotcunque factoribus aequalibus vti ex §. 1120. colligere licet.

Coroll. 3.

Coroll. 3.

1128. Si factores occurrant imaginarii, bini coniuncti exhibent factorem trinomium realem, cuius forma ita repraesentatur

$$ff + 2fg\lambda\cos.\zeta + gg\lambda\lambda,$$

vnde deducitur

$$\lambda = -\frac{f}{g}(\cos.\zeta \pm \sqrt{-1}.\sin.\zeta),$$

quo cum §. 1121. collato fit $\mu = \frac{-f\cos.\zeta}{g}$ et $\nu = \frac{f\sin.\zeta}{g}$. Ex quo pars integralis ex tali factore oriunda erit

$$e^{\frac{-fx\cos.\zeta}{g}}(\mathfrak{A}\cos.\frac{fx\sin.\zeta}{g} + \mathfrak{B}\sin.\frac{fx\sin.\zeta}{g}).$$

Coroll. 4.

1129. Si huiusmodi formae quadratum inter factores occurrat

$$(ff + 2fg\lambda\cos.\zeta + gg\lambda\lambda)^2,$$

seu duo huiusmodi factores sint aequales, considerentur quasi infinite parum discrepantes, vt in altero loco $\frac{f}{g}$ fit $\frac{f}{g}(1+\omega)$; et ob

$$e^{\frac{-fx\cos.\zeta}{g}(1+\omega)} = e^{\frac{-fx\cos.\zeta}{g}}(1 - \frac{\omega fx}{g}\cos.\zeta);$$

$$\cos.\frac{fx\sin.\zeta}{g}(1+\omega) = \cos.\frac{fx\sin.\zeta}{g} - \frac{\omega fx\sin.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g} \text{ et}$$

$$\sin.\frac{fx\sin.\zeta}{g}(1+\omega) = \sin.\frac{fx\sin.\zeta}{g} + \frac{\omega fx\sin.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}:$$

ex hoc factore colligitur pars integralis

$$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{array}{l}\mathfrak{A}'\cos.\frac{fx\sin.\zeta}{g}-\mathfrak{A}'\frac{\omega fx\cos.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}-\mathfrak{A}'\frac{\omega fx\sin.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g}\\+\mathfrak{B}'\sin.\frac{fx\sin.\zeta}{g}-\mathfrak{B}'\frac{\omega fx\cos.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g}+\mathfrak{B}'\frac{\omega fx\sin.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}\end{array}\right\}$$

cui prior addi debet. Hunc in finem constantes ita contrahamus ponendo

$$\mathfrak{A}+\mathfrak{A}'=\mathfrak{E};\quad \frac{-\mathfrak{A}'\omega f\cos.\zeta}{g}+\frac{\mathfrak{B}'\omega f\sin.\zeta}{g}=\mathfrak{G}$$

$$\mathfrak{B}+\mathfrak{B}'=\mathfrak{F};\quad \frac{-\mathfrak{A}'\omega f\sin.\zeta}{g}-\frac{\mathfrak{B}'\omega f\cos.\zeta}{g}=\mathfrak{H}$$

vnde illae constantes vtique determinantur, eritque pars integralis respondens:

$$e^{\frac{-fx\cos.\zeta}{g}}((\mathfrak{E}+\mathfrak{G}x)\cos.\frac{fx\sin.\zeta}{g}+(\mathfrak{F}+\mathfrak{H}x)\sin.\frac{fx\sin.\zeta}{g}.$$

Scholion.

1130. En ergo vniuersam methodum huiusmodi aequationum differentialium integralia inueniendi

$$Ay+B\frac{dy}{dx}+C\frac{ddy}{dx^2}+D\frac{d^3y}{dx^3}+\ldots\ldots+N\frac{d^ny}{dx^n}=0$$

ita in compendium contractam. Scribatur vt iste laterculus indicat:

loco y	$\frac{dy}{dx}$	$\frac{ddy}{dx^2}$	$\frac{d^3y}{dx^3}$	$\frac{d^4y}{dx^4}$	.....	$\frac{d^ny}{dx^n}$
scribatur 1	z	z^2	z^3	z^4		z^n

vt oriatur haec aequatio algebraica

$$A+Bz+Cz^2+Dz^3+Ez^4+\ldots\ldots+Nz^n=0$$

cuius

cuius ſinguli factores reales, ſiue ſimplices ſiue duplicati notentur, atque inſuper caſus quibus duo pluresue ſunt inter ſe aequales probe obſeruentur. Tum cuiusmodi partes pro Integrali quaeſito ex ſingulis factoribus oriantur, ex ſequente Tabella intelligere licet:

Factores	Partes integralis
$f+gz$	$\mathfrak{A}e^{\frac{-fx}{g}}$
$(f+gz)^2$	$(\mathfrak{A}+\mathfrak{B}x)e^{\frac{-fx}{g}}$
$(f+gz)^3$	$(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2)e^{\frac{-fx}{g}}$
$(f+gz)^4$	$(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3)e^{\frac{-fx}{g}}$
$ff+2fgz\cos.\zeta+ggzz$	$e^{\frac{-fx\cos.\zeta}{g}}(\mathfrak{A}\cos.\frac{fx\sin.\zeta}{g}+\mathfrak{B}\sin.\frac{fx\sin.\zeta}{g})$
$(ff+2fgz\cos.\zeta+ggzz)^2$	$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{array}{l}(\mathfrak{A}+\mathfrak{B}x)\cos.\frac{fx\sin.\zeta}{g}\\(\mathfrak{a}+\mathfrak{b}x)\sin\frac{fx\sin.\zeta}{g}\end{array}\right.$
$(ff+2fgz\cos.\zeta+ggzz)^3$	$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{array}{l}\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}xx)\cos.\frac{fx\sin.\zeta}{g}\\(\mathfrak{a}+\mathfrak{b}x+\mathfrak{c}xx)\sin.\frac{fx\sin.\zeta}{g}\end{array}\right.$
	etc.

Pro ſingulis autem factoribus diuerſae litterae conſtantes ſcribi debent, vt integrale omnibus numeris completum obtineatur.

Exemplum 1.

1131. *Aequationis differentialis quarti gradus:*

$$y-\frac{2dy}{dx}+\frac{2ddy}{dx^2}-\frac{2d^3y}{dx^3}+\frac{d^4y}{dx^4}=0$$

integrale completum aſſignare.

 Hinc

Hinc oritur aequatio algebraica:

$$1-2z+2zz-2z^3+z^4=0$$

quae in hos factores resoluitur $(1-z)^2(1+zz)$, quorum prior ob $f=1$ et $g=-1$ praebet hanc partem integralis $(\mathfrak{A}+\mathfrak{B}x)e^x$, alter vero factor ob $f=1$; $\cos.\zeta=0$; $g=1$ et $\sin.\zeta=1$ dat: $\mathfrak{A}\cos.x+\mathfrak{B}\sin.x$. Quare integrale completum, quod quaeritur, erit

$$y=(\mathfrak{A}+\mathfrak{B}x)e^x+\mathfrak{C}\cos.x+\mathfrak{D}\sin.x$$

continens quatuor constantes arbitrarias. Quod si velimus vt posito $x=0$, fiat $y=0$, fieri oportet $\mathfrak{A}+\mathfrak{C}=0$, si etiam $\frac{dy}{dx}$ eodem casu euanescere debeat, ob

$$\frac{dy}{dx}=(\mathfrak{A}+\mathfrak{B}+\mathfrak{B}x)e^x-\mathfrak{C}\sin.x+\mathfrak{D}\cos.x$$

fieri debet $\mathfrak{A}+\mathfrak{B}+\mathfrak{D}=0$. Si praetera $\frac{ddy}{dx^2}$ euanescere debeat, ob

$$\frac{ddy}{dx^2}=(\mathfrak{A}+2\mathfrak{B}+\mathfrak{B}x)e^x-\mathfrak{C}\cos.x-\mathfrak{D}\sin.x$$

fieri debet $\mathfrak{A}+2\mathfrak{B}-\mathfrak{C}=0$. Quare his tribus conditionibus satisfaciemus sumendo $\mathfrak{C}=-\mathfrak{A}$; $\mathfrak{B}=-\mathfrak{A}$ et $\mathfrak{D}=0$, ita vt sit integrale:

$$y=\mathfrak{A}(1-x)e^x-\mathfrak{A}\cos.x.$$

Exemplum 2.

1132. *Aequationem differentialem quarti ordinis*

$$Ay+C\frac{ddy}{dx^2}+E\frac{d^4y}{dx^4}=0$$

sumto elemento dx *constante integrare.*

Aequa-

Aequatio algebraica ad integrationem perducens eſt:

$$A+Czz+Ez^4=0$$

quae ſemper duos factores duplicatos reales habet, quorum forma duplex eſſe poteſt:

vel $(aa+2maz+nzz)(aa-2maz+nzz)$

vel $(aa+mzz)(aa+nzz)$.

Ex priori eſt:

$$A=a^4;\ C=2naa-4mmaa;\ E=nn$$

ex poſteriore vero

$$A=a^4;\ C=(m+n)aa;\ E=mn$$

ſemper autem terminum primum A biquadrato a^4 repraeſentare licet, et prior reſolutio locum habet, ſi E ſit numerus poſitiuus, et $2naa-C$ ſeu $2\sqrt{AE}-C$ quoque poſitiuus, ideoque $4AE>CC$. Poſterior vero ſi $CC<4AE$. Tum igitur videndum eſt, ad quamnam claſſem ſinguli factores pertineant, vnde ſequentes caſus occurrent:

I. Si omnes quatuor factores ſimplices ſunt reales, erit

$$A+Czz+Ez^4=(a+z)(a-z)(b+z)(b-z)$$

haec habebitur aequatio:

$$aabby-(aa+bb)\frac{ddy}{dx^2}+\frac{d^4y}{dx^4}=0$$

cuius integrale completum eſt:

$$y=\mathfrak{A}e^{ax}+\mathfrak{B}e^{-ax}+\mathfrak{C}e^{bx}+\mathfrak{D}e^{-bx}.$$

Ac

Ac si sit $b=a$, huius aequationis

$$a^4y-\frac{2aaddy}{dx^2}+\frac{d^4y}{dx^4}=0$$

integrale completum erit

$$y=(\mathfrak{A}+\mathfrak{B}x)e^{ax}+(\mathfrak{C}+\mathfrak{D}x)e^{-ax}.$$

II Si duo factores simplices sint reales, duo vero imaginarii, erit

$$A+Czz+Ez^4=(a+z)(a-z)(bb+zz)$$

haec habebitur aequatio:

$$aabby+(aa-bb)\frac{ddy}{dx^2}-\frac{d^4y}{dx^4}=0$$

cuius integrale completum est:

$$y=\mathfrak{A}e^{ax}+\mathfrak{B}e^{-ax}+\mathfrak{C}\operatorname{cof.}bx+\mathfrak{D}\operatorname{fin.}bx.$$

III. Si omnes factores simplices sint imaginarii, duo casus sunt euoluendi:

1) si $A+Czz+Ez^4=(aa+zz)(bb+zz)$

vnde huius aequationis

$$aabby+(aa+bb)\frac{ddy}{dx^2}+\frac{d^4y}{dx^4}=0$$

integrale completum erit:

$$y=\mathfrak{A}\operatorname{cof.}ax+\mathfrak{B}\operatorname{fin.}ax+\mathfrak{C}\operatorname{cof.}bx+\mathfrak{D}\operatorname{fin.}bx$$

2) si $A+Czz+Ez^4=(aa+2az\operatorname{cof}\zeta+zz)(aa-2az\operatorname{cof.}\zeta+zz)$.

vnde huius aequationis

$$a^4y-\frac{2aaddy}{dx^2}\operatorname{cof.}2\zeta+\frac{d^4y}{dx^4}=0$$

inte-

integrale completum eſt

$$y = e^{+ax\cos.\zeta}(\mathfrak{A}\cos.ax\sin.\zeta + \mathfrak{B}\sin.ax\sin.\zeta)$$
$$+ e^{-ax\cos.\zeta}(\mathfrak{C}\cos.ax\sin.\zeta + \mathfrak{D}\sin.ax\sin.\zeta).$$

At ſi ſit priori caſu $b = a$ ſeu poſteriori $\cos.\zeta = 0$ huius aequationis

$$a^4y + \frac{2aaddy}{dx^2} + \frac{d^4y}{dx^4} = 0$$

integrale completum eſt

$$y = (\mathfrak{A} + \mathfrak{B}x)\cos.ax + (\mathfrak{C} + \mathfrak{D}x)\sin.ax$$

Scholion 1.

1133. Cum igitur aequationis

$$Ay + \frac{Cddy}{dx^2} + \frac{Ed^4y}{dx^4} = 0$$

integrale aſſignari poſſit, omnes aequationes quas inde deriuare licet, integrari poterunt. At haec aequatio per $2dy$ multiplicata primo per integrationem ad differentialem tertii ordinis reducitur:

$$Ayy + \frac{Cdy^2}{dx^2} + \frac{2Edyd^3y - Eddy^2}{dx^4} = \text{Conſt.}$$

In integrali autem ante inuento conſtantes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ ita definire licet, vt haec conſtans Conſt. euaneſcat, ideoque huius aequationis

$$Ayy + \frac{Cdy^2}{dx^2} + \frac{2Edyd^3y - Eddy^2}{dx^4} = 0$$

integrale completum erit in noſtra poteſtate. Nunc ponatur $y = e^{\int vdx}$ vt ſit $v = \frac{dy}{ydx}$, et ob $\frac{dy}{dx} = e^{\int vdx}v$; $\frac{ddy}{dx^2} = e^{\int vdx}(\frac{dv}{dx} + vv)$ atque $\frac{d^3y}{dx^3} = e^{\int vdx}(\frac{ddv}{dx^2} + \frac{3vdv}{dx} + v^3)$,

 aequa-

aequatio noſtra hanc induit formam:

$$A+Cvv+E\left(\frac{2vddv}{dx^2}+\frac{4vvdv}{dx}+v^4-\frac{dv^2}{dx^2}\right)=0.$$

Sit porro $dx=\frac{dv}{s}$, vt ſit $s=\frac{dv}{dx}=\frac{ddy}{ydx^2}-\frac{dy^2}{yydx^2}$ erit $\frac{ddv}{dx}=ds$ et $\frac{ddv}{dx^2}=\frac{sds}{dv}$, vnde reſultat haec aequatio differentialis primi gradus:

$$A+Cvv+E\left(\frac{2vsds}{dv}-ss+4vvs+v^4\right)=0$$

cuius relatio inter v et s ita ex relatione inter x et y inuenta definitur, vt ſit

$$v=\frac{dy}{ydx} \text{ et } s=\frac{yddy-dy^2}{yydx^2}.$$

Scholion 2.

1134. Retenta autem illa conſtante per integrationem ingreſſa, vt habeatur:

$$Ayy+\frac{Cdy^2}{dx^2}+\frac{2Edyd^3y-Eddy^2}{dx^4}=G$$

in integrali completo, quo y per x exprimitur, conſtantes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ quantitati huic G conformiter determinari poterunt. Nunc igitur ponatur $dx=\frac{dy}{u}$, vt ſit $\frac{dy}{dx}=u$, erit $\frac{ddy}{dx^2}=\frac{udu}{dy}$, et $\frac{d^3y}{dx^3}=d.\frac{udu}{dy}$ ideoque $\frac{d^3y}{dx^3}=\frac{u}{dy}d.\frac{udu}{dy}$. Vnde obtinetur haec aequatio differentialis ſecundi gradus:

$$Ayy+Cuu+E\left(\frac{2uu}{dy}d.\frac{udu}{dy}-\frac{uudu^2}{dy^2}\right)=G$$

vbi conſideratio elementi pro conſtante aſſumti eſt exuta. Nihil ergo impedit quo minus ſumamus dy pro

pro conſtante, fietque

$$Ayy + Cuu + E\left(\frac{2u^3ddu}{dy^2} + \frac{uudu^2}{dy^2}\right) = G$$

quae ergo aequatio etiam integrari poteſt.

Vel ſi ponamus $yy = p$ et $uu = q$, ſumto elemento dp conſtante, prodibit haec aequatio:

$$Ap + Cq + E\left(\frac{4pqddq + 2qdpdq - pdq^2}{dp^2}\right) = G.$$

Vel ſi in illa aequatione ponatur $u = r^{\frac{2}{3}}$, erit

$$Ayy + Cr^{\frac{4}{3}} + \tfrac{4}{3}Er^{\frac{5}{3}}.\frac{ddr}{dy^2} = G.$$

Quarum formarum integratio ſine hoc ſubſidio maxime ardua videtur.

Problema 146.

1135. Propoſita aequatione differentiali ordinis cuiuscunque $a^n y \pm \frac{d^n y}{dx^n} = 0$, vbi elementum dx conſtans eſt aſſumtum, eius integrale completum inveſtigare.

Solutio.

Aequatio algebraica ſolutioni inſeruiens, eſt $a^n \pm z^n = 0$ pro cuius reſolutione duos caſus conſiderari conuenit, prout ſignum vel ſuperius vel inferius valeat.

I. Valeat ſuperius, vt haec propoſita ſit aequatio

$$a^n y + \frac{d^n y}{dx^n} = 0$$

et formulae $a^n + z^n$ factores reales sunt

$$aa - 2az\cos.\tfrac{\pi}{n} + zz;\quad aa - 2az\cos.\tfrac{3\pi}{n} + zz;$$
$$aa - 2az\cos.\tfrac{5\pi}{n} + zz;\ \text{etc.}$$

quorum vltimus est vel $aa - 2az\cos.\frac{n\pi}{n} + zz$ vel $aa - 2az\cos.\frac{(n-1)\pi}{n} + zz$, prout vel n vel $n-1$ fuerit numerus impar, atque illo quidem casu loco factoris quadrati $aa + 2az + zz$ eius radix $a + z$ sumi debet.

Hinc istius aequationis integrale completum est

$$y = + e^{ax\cos.\frac{\pi}{n}}\left(\mathfrak{A}\cos.ax\sin.\tfrac{\pi}{n} + \mathfrak{B}\sin.ax\sin.\tfrac{\pi}{n}\right)$$
$$+ e^{ax\cos.\frac{3\pi}{n}}\left(\mathfrak{C}\cos.ax\sin.\tfrac{3\pi}{n} + \mathfrak{D}\sin.ax\sin.\tfrac{3\pi}{n}\right)$$
$$+ e^{ax\cos.\frac{5\pi}{n}}\left(\mathfrak{E}\cos.ax\sin.\tfrac{5\pi}{n} + \mathfrak{F}\sin.ax\sin.\tfrac{5\pi}{n}\right)$$
$$\text{etc.}$$

cuius expressionis, si n sit numerus impar, vltima pars sit $\mathfrak{N}e^{-ax}$. Quod integrale etiam ita potest exhiberi

$$y = \mathfrak{A}e^{ax\cos.\frac{\pi}{n}}\cos.(ax\sin.\tfrac{\pi}{n} + \mathfrak{a}) + \mathfrak{B}e^{ax\cos.\frac{3\pi}{n}}\cos.(ax\sin.\tfrac{3\pi}{n} + \mathfrak{b})$$
$$+ \mathfrak{C}e^{ax\cos.\frac{5\pi}{n}}\cos.(ax\sin.\tfrac{5\pi}{n} + \mathfrak{c}) + \mathfrak{D}e^{ax\cos.\frac{7\pi}{n}}\cos.(ax\sin.\tfrac{7\pi}{n} + \mathfrak{d})$$
$$\text{etc.}$$

quae forma eousque continuari debet, quoad similes termini recurrant.

II.

II. Si valeat signum inferius, propositaque sit haec aequatio:

$$a^n y - \frac{d^n y}{d x^n} = 0$$

formulae $a^n - z^n$ factores reales sunt:

$$a - z;\ aa - 2az\cos.\tfrac{2\pi}{n} + zz;\ aa - 2az\cos.\tfrac{4\pi}{n} + zz;$$
$$aa - 2az\cos.\tfrac{6\pi}{n} + zz;\ \text{etc.}$$

quorum si n numerus par, vltimus est $a+z$, sin autem impar

$$aa - 2az\cos.\tfrac{(n-1)\pi}{n} + zz.$$

Quare aequationis huius integrale completum est

$$y = \mathfrak{A}e^{ax} + e^{ax\cos.\frac{2\pi}{n}}(\mathfrak{B}\cos.ax\sin.\tfrac{2\pi}{n} + \mathfrak{C}\sin.ax\sin.\tfrac{2\pi}{n})$$
$$+ e^{ax\cos.\frac{4\pi}{n}}(\mathfrak{D}\cos.ax\sin.\tfrac{4\pi}{n} + \mathfrak{E}\sin.ax\sin.\tfrac{4\pi}{n})$$
$$+ e^{ax\cos.\frac{6\pi}{n}}(\mathfrak{F}\cos.ax\sin.\tfrac{6\pi}{n} + \mathfrak{G}\sin.ax\sin.\tfrac{6\pi}{n})$$

etc.

quod integrale etiam ita exprimi potest:

$$y = \mathfrak{A}e^{ax} + \mathfrak{B}e^{ax\cos.\frac{2\pi}{n}}\cos.(ax\sin.\tfrac{2\pi}{n} + \mathfrak{b})$$
$$+ \mathfrak{C}e^{ax\cos.\frac{4\pi}{n}}\cos.(ax\sin.\tfrac{4\pi}{n} + \mathfrak{c}) + \mathfrak{D}e^{ax\cos.\frac{6\pi}{n}}\cos.(ax\sin.\tfrac{6\pi}{n} + \mathfrak{d})$$

etc.

quae forma eousque est continuanda, quamdiu termini a prioribus diuersi prodeunt.

Scholion 1.

1136. Pro variis ergo exponentis n valoribus integralia ſequenti modo ſe habebunt, ac primo quidem pro aequatione $a^n y + \frac{d^n y}{dx^n} = 0$.

I. Aequationis $ay + \frac{dy}{dx} = 0$ integrale eſt $y = \mathfrak{A} e^{-ax}$

II. Aequationis $a^2 y + \frac{ddy}{dx^2} = 0$ integrale eſt $y = \mathfrak{A} \operatorname{cof.} (ax + \mathfrak{a})$.

III. Aequationis $a^3 y + \frac{d^3 y}{dx^3} = 0$ integrale eſt

$$y = \mathfrak{A} e^{\frac{1}{2} ax} \operatorname{cof.} \left(\frac{ax\sqrt{3}}{2} + \mathfrak{a}\right) + \mathfrak{B} e^{-ax}.$$

IV. Aequationis $a^4 y + \frac{d^4 y}{dx^4} = 0$ integrale eſt

$$y = \mathfrak{A} e^{\frac{ax}{\sqrt{2}}} \operatorname{cof.} \left(\frac{ax}{\sqrt{2}} + \mathfrak{a}\right) + \mathfrak{B} e^{\frac{-ax}{\sqrt{2}}} \operatorname{cof.} \left(\frac{ax}{\sqrt{2}} + \mathfrak{b}\right).$$

V. Aequationis $a^5 y + \frac{d^5 y}{dx^5} = 0$ integrale eſt

$$y = \mathfrak{A} e^{ax \operatorname{cof.} 36^\circ} \operatorname{cof.} (ax \operatorname{fin.} 36^\circ + \mathfrak{a}) + \mathfrak{B} e^{-ax \operatorname{cof.} 72^\circ} \operatorname{cof.} (ax \operatorname{fin.} 72^\circ + \mathfrak{b}) + \mathfrak{C} e^{-ax}.$$

VI. Aequationis $a^6 y + \frac{d^6 y}{dx^6} = 0$ integrale eſt

$$y = \mathfrak{A} e^{\frac{ax\sqrt{3}}{2}} \operatorname{cof.} \left(\tfrac{1}{2} ax + \mathfrak{a}\right) + \mathfrak{B} \operatorname{cof.} (ax + \mathfrak{b}) + \mathfrak{C} e^{\frac{-ax\sqrt{3}}{2}} \operatorname{cof.} \left(\tfrac{1}{2} ax + \mathfrak{c}\right)$$

etc.

Simili autem modo pro altera forma

$$a^n y - \frac{d^n y}{dx^n} = 0$$

inte-

integrationes ad valores fimpliciores exponentis n accommodatae ita fe habebunt:

I. Aequationis $ay-\frac{dy}{dx}=0$ integrale eft:

$$y=\mathfrak{A}e^{ax}.$$

II. Aequationis $a^2y-\frac{ddy}{dx^2}=0$ integrale eft:

$$y=\mathfrak{A}e^{ax}+\mathfrak{B}e^{-ax}.$$

III. Aequationis $a^3y-\frac{d^3y}{dx^3}=0$ integrale eft:

$$y=\mathfrak{A}e^{ax}+\mathfrak{B}e^{-\frac{1}{2}ax}\operatorname{cof.}\left(\frac{ax\sqrt{3}}{2}+\mathfrak{b}\right)$$

IV. Aequationis $a^4y-\frac{d^4y}{dx^4}=0$ integrale eft:

$$y=\mathfrak{A}e^{ax}+\mathfrak{B}\operatorname{cof.}(ax+\mathfrak{b})+\mathfrak{C}e^{-ax}.$$

V. Aequationis $a^5y-\frac{d^5y}{dx^5}=0$ integrale eft:

$$y=\mathfrak{A}e^{ax}+\mathfrak{B}e^{ax\operatorname{cof.}72^\circ}\operatorname{cof.}(ax\operatorname{fin.}72^\circ+\mathfrak{b})$$
$$+\mathfrak{C}e^{-ax\operatorname{cof.}36^\circ}\operatorname{cof.}(ax\operatorname{fin.}36^\circ+\mathfrak{c}).$$

VI. Aequationis $a^6y-\frac{d^6y}{dx^6}=0$ integrale eft:

$$y=\mathfrak{A}e^{ax}+\mathfrak{B}e^{\frac{1}{2}ax}\operatorname{cof.}\left(\frac{ax\sqrt{3}}{2}+\mathfrak{b}\right)+\mathfrak{C}e^{-\frac{1}{2}ax}\operatorname{cof.}\left(\frac{ax\sqrt{3}}{2}+\mathfrak{c}\right)+\mathfrak{D}e^{-ax}$$

ficque quousque libuerit progredi licet.

Scho-

Scholion 2.

1137. Quamuis methodus, qua hic ſum vſus, expedite integralia aequationum in propoſita forma contentarum ſuppeditet, a principiis tamen integrationis omnino abhorret. Cum enim aequatio differentialis eſt altioris gradus, leges integrationis poſtulant, vt toties ſeorſim integretur, antequam ad relationem finitam inter binas variabiles perueniatur, et dum ſingulae integrationes conſtantem arbitrariam recipiunt, hoc demum modo integrale completum obtinetur. Hactenus autem vna quaſi operatione integrale poſtremum eruimus, cum omnibus conſtantibus, quibus id completum redditur; reuera ſcilicet ſola coniectura vtentes plura integralia particularia ſumus adepti, atque natura aequationis commode permiſit, vt ex iis integrale completum formare liceret. Verum ſi leges integrandi ſtricte obſeruare velimus, propoſita verbi gratia aequatione differentiali quarti gradus, quadruplici integratione opus erit, quarum prima, ea reducatur ad aequationem differentialem tertii gradus, tum vero iſta per nouam integrationem ad aequationem differentialem ſecundi gradus, quae denuo integrata ad gradum primum perducatur, haecque tandem iterum integrata relationem quaeſitam inter binas variabiles patefaciat. Atque hoc modo etiam aequationum hic tractatarum formam reſoluere licet, vt per continuas integrationes ad gradus ſimpliciores redigatur, quibus tandem eadem integralia, quae hic elicuimus, inueniantur.

Cum

Cum autem haec methodus latius pateat, quam ad formas hic consideratas, eiusque ope haec aequatio generalior integrari queat:

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \text{etc.}$$

denotante X functionem quamcunque ipsius x, cui resoluendae praecedens methodus minime sufficit, nouam methodum statim ad hanc formam generaliorem accommodabo.

CAPVT III.

DE INTEGRATIONE AEQVATIONVM DIFFERENTIALIVM HVIVS FORMAE

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \text{etc.}$$

Problema 147.

1138.

Proposita aequatione differentiali

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \text{etc.} \ldots + \frac{Nd^ny}{dx^n}$$

sumto elemento dx constante, et significante X functionem quamcunque ipsius x, inuenire functionem ipsius x, per quam haec aequatio multiplicata integrabilis euadat.

Solutio.

Sit Pdx iste multiplicator quem quaerimus, et cum prius membrum X eo integrabile reddatur, eius rationem ex altero membro definiri oportet. Facile autem intelligitur, formam huius multiplicatoris P eiusmodi fore $e^{\lambda x}$, ita vt quantitas λ definiri

finiri debeat. Sit ergo $e^{\lambda x}dx$ multiplicator, atque hanc formam:

$$e^{\lambda x}dx\left(Ay+\frac{Bdy}{dx}+\frac{Cddy}{dx^2}+\frac{Dd^3y}{dx^3}+\ldots\ldots+\frac{Nd^ny}{dx^n}\right)$$

integrabilem esse oportet, cuius integrale propterea statuatur

$$e^{\lambda x}\left(A'y+\frac{B'dy}{dx}+\frac{C'ddy}{dx^2}+\ldots\ldots+\frac{M'd^{n-1}y}{dx^{n-1}}\right)$$

ita vt huius differentiale cum illa forma congruere debeat, quod cum sit

$$e^{\lambda x}dx\left\{\begin{array}{l}\lambda A'y+\frac{\lambda B'dy}{dx}+\frac{\lambda C'ddy}{dx^2}+\ldots\ldots+\frac{\lambda M'd^{n-1}y}{dx^{n-1}}\\ \qquad+\frac{A'dy}{dx}+\frac{B'ddy}{dx^2}+\ldots\ldots\ldots\ldots\ldots\ldots+\frac{M'd^ny}{dx^n}\end{array}\right\}$$

necesse est sit:

$A'=\frac{A}{\lambda}$; $B'=\frac{B-A'}{\lambda}$; $C'=\frac{C-B'}{\lambda}$; $D'=\frac{D-C'}{\lambda}$;
$M'=\frac{M-L'}{\lambda}$;

atque $M'=N$. Hinc erit:

$$A'=\frac{A}{\lambda}$$
$$B'=\frac{B}{\lambda}-\frac{A}{\lambda^2}$$
$$C'=\frac{C}{\lambda}-\frac{B}{\lambda^2}+\frac{A}{\lambda^3}$$
$$D'=\frac{D}{\lambda}-\frac{C}{\lambda^2}+\frac{B}{\lambda^3}-\frac{A}{\lambda^4}$$
$$\vdots$$
$$M'=\frac{M}{\lambda}-\frac{L}{\lambda^2}+\frac{K}{\lambda^3}\ldots\ldots\pm\frac{A}{\lambda^n} \text{ et}$$
$$0=\frac{N}{\lambda}-\frac{M}{\lambda^2}+\frac{L}{\lambda^3}\ldots\ldots\mp\frac{A}{\lambda^{n+1}}$$

vbi ex vltima aequatione quantitas λ erui debet, quae aequatio induit hanc formam:

$$A-B\lambda+C\lambda^2-D\lambda^3+E\lambda^4 \ldots\ldots \pm N\lambda^n=0$$

vnde cum λ sortiatur n valores, totidem quoque multiplicatores inueniuntur.

Videamus, quomodo hae determinationes pro singulis valoribus exponentis n se habeant.

I. Si $n=1$; erit $A-B\lambda=0$, tum vero

$A'=\frac{A}{\lambda}=B.$

II. Si $n=2$; erit $A-B\lambda+C\lambda^2=0$, tum vero

$A'=\frac{A}{\lambda}=B-C\lambda$ et $B'=\frac{B\lambda-A}{\lambda^2}=C.$

III. Si $n=3$; erit $A-B\lambda+C\lambda^2-D\lambda^3=0$ tum vero

$A'=\frac{A}{\lambda}=B-C\lambda+D\lambda^2$;

$B'=\frac{B\lambda-A}{\lambda^2}=C-D\lambda$ et

$C'=\frac{C\lambda^2-B\lambda+A}{\lambda^3}=D.$

IV. Si $n=4$; erit $A-B\lambda+C\lambda^2-D\lambda^3+E\lambda^4=0$; tum vero

$A'=\frac{A}{\lambda}=B-C\lambda+D\lambda^2-E\lambda^3$

$B'=\frac{B\lambda-A}{\lambda^2}=C-D\lambda+E\lambda^2$

$C'=\frac{C\lambda^2-B\lambda+A}{\lambda^3}=D-E\lambda$

$D'=\frac{D\lambda^3-C\lambda^2+B\lambda-A}{\lambda^4}=E.$

V. Si

V. Si $n=5$; erit $A-B\lambda+C\lambda^2-D\lambda^3+E\lambda^4-F\lambda^5=0$; tum vero

$$A'=\frac{A}{\lambda}=B-C\lambda+D\lambda^2-E\lambda^3+F\lambda^4$$

$$B'=\frac{B\lambda-A}{\lambda^2}=C-D\lambda+E\lambda^2-F\lambda^3$$

$$C'=\frac{C\lambda^2-B\lambda+A}{\lambda^3}=D-E\lambda+F\lambda^2$$

$$D'=\frac{D\lambda^3-C\lambda^2+B\lambda-A}{\lambda^4}=E-F\lambda$$

$$E'=\frac{E\lambda^4-D\lambda^3+C\lambda^2-B\lambda+A}{\lambda^5}=F$$

atque ita porro.

Inuento autem hoc multiplicatore $e^{\lambda x}dx$ prius aequationis membrum fit $\int e^{\lambda x}X\,dx$, et aequatio proposita, quae est differentialis gradus n per integrationem reducitur ad hanc vno gradu simpliciorem:

$$\int e^{\lambda x}X\,dx=e^{\lambda x}\left(A'y+B'\frac{dy}{dx}+C'\frac{ddy}{dx^2}+\ldots\ldots+M'\frac{d^{n-1}y}{dx^{n-1}}\right).$$

Coroll. 1.

1139. Integratione ergo hac prima instituta, aequatio proposita vno gradu deprimitur, et definitis coefficientibus A', B', C' etc. ex superioribus formulis aequatio integralis hac forma exhiberi potest:

$$e^{-\lambda x}\int e^{\lambda x}X\,dx=A'y+B'\frac{dy}{dx}+C'\frac{ddy}{dx^2}+\ldots\ldots+M'\frac{d^{n-1}y}{dx^{n-1}}.$$

Coroll. 2.

1140. Cum prius membrum $e^{-\lambda x}\int e^{\lambda x}X\,dx$ ſit functio ipſius x conſtantem arbitrariam inuoluens, ſi eius loco ponatur X', haec aequatio ſimilem formam habet atque ipſa propoſita, ideoque eadem methodo iterum integrari et ad gradum differentialitatis $n-2$ reduci poteſt, quae huiusmodi formam habebit:

$$X'' = A''y + B''\frac{dy}{dx} + C''\frac{ddy}{dx^2} + \dots + L''\frac{d^{n-2}y}{dx^{n-2}}.$$

Coroll. 3.

1141. Hoc modo vlterius progrediendo tandem ad aequationem differentialem primi gradus peruenietur $X^{(n-1)} = A^{(n-1)}y + B^{(n-1)}\frac{dy}{dx}$, quae ſimili modo ad aequationem finitam $X^{(n)} = A^{(n)}y$, reducitur, qua relatio inter ipſas variabiles x et y exprimitur.

Scholion.

1142. Haec igitur eſt methodus huiusmodi aequationes differentiales altiorum graduum ſucceſſiue per gradus integrandi, vbi tot opus eſt integrationibus, quoti gradus differentialis fuerit ipſa aequatio propoſita. Totum ergo negotium ſitum eſt in inuentione ſucceſſiua coefficientium, quos ex praecedentibus ope multiplicatoris definiri oportet. In genere quidem lex, qua ii continuo ex antecedentibus deter-

determinantur, non ita eſt perſpicua, vt inde forma integrabilis extremi perſpici poſſit, verum quia ex capite ſuperiori nouimus caſu, quo primum membrum X euaneſcit, etiam vltimum integrale lege ſatis ſimplici contineri, idem hic vſu venire merito ſuſpicamur; eamque legem facillime agnoſcemus, ſi pedetentim a gradibus inferioribus ad altiores progrediamur. Ac primo quidem caſu, quo aequatio eſt differentialis primi gradus $X = Ay + B\frac{dy}{dx}$, multiplicator erit $e^{\lambda x}dx$ poſito $A - \lambda B = o$, vt ſit $\lambda = \frac{A}{B}$; et cum ſit $A' = \frac{A}{\lambda} = B$, integrale erit

$$\int e^{\lambda x}Xdx = Be^{\lambda x}y \text{ ſeu } e^{-\lambda x}\int e^{\lambda x}Xdx = By.$$

Ad hanc ſimilitudinem aequationes graduum altiorum euoluamus, ac formam integralis vltimi inueſtigemus.

Problema 148.

1143. Propoſita aequatione differentiali ſecundi gradus

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2}$$

per duplicem integrationem relationem inter x et y inueſtigare.

Solutio.

Sit $e^{\lambda x}dx$ multiplicator hanc aequationem per ſe integrabilem reddens, eritque $A - B\lambda + C\lambda^2 = o$, tum ſumatur

$$A' = \frac{A}{\lambda} = B - C\lambda \text{ et } B' = \frac{B\lambda - A}{\lambda^2} = C,$$

poſito

positoque

$$e^{-\lambda x}\int e^{\lambda x} X\,dx = X',$$

aequatio semel integrata est

$$X' = A'y + B'\frac{dy}{dx}.$$

Huius iam multiplicator sit $e^{\mu x}dx$, eritque: $A' - B'\mu = 0$, ac statuatur $A'' = \frac{A'}{\mu} = B'$; positoque $e^{-\mu x}\int e^{\mu x} X' dx = X''$ habebimus $X'' = A''y$, quae est aequatio bis integrata relationem quaesitam inter x et y exprimens.

Cum igitur hic sit $A'' = B'$ et $B' = C$ erit $A'' = C$. Deinde loco A' et B' substitutis valoribus, aequatio $A' - B'\mu = 0$ induit hanc formam:

$$B - C\lambda - C\mu = 0 \text{ seu } B - C(\lambda + \mu) = 0,$$

ex qua cum sit $\lambda + \mu = \frac{B}{C}$, patet $\lambda + \mu$ aequari summae binarum radicum aequationis $A - B\lambda + C\lambda^2 = 0$. Quoniam igitur λ eius vna est radix, μ necessario eius alteram radicem denotat. Quare si ex aequatione proposita, vti in capite praecedente fecimus, hanc formemus aequationem $A + Bz + Czz = 0$, eius radices erunt $z = -\lambda$ et $z = -\mu$. Seu si factores eius statuamus $C(\alpha + z)(\beta + z)$ litterae α et β praebebunt valores λ et μ. Hinc cum sit

$$X' = e^{-\alpha x}\int e^{\alpha x} X\,dx \text{ erit}$$

$$X'' = e^{-\beta x}\int e^{(\beta-\alpha)x}dx\int e^{\alpha x} X\,dx.$$

At $\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}X\,dx = \frac{1}{\beta-\alpha}e^{(\beta-\alpha)x}\int e^{\alpha x}X\,dx - \frac{1}{\beta-\alpha}\int e^{\beta x}X\,dx$

vnde

vnde concluditur :

$$X'' = \frac{1}{\beta-\alpha} e^{-\alpha x} \int e^{\alpha x} X dx + \frac{1}{\alpha-\beta} e^{-\beta x} \int e^{\beta x} X dx$$

Quocirca aequationis propofitae integrale completum eft

$$Cy = \frac{1}{\beta-\alpha} e^{-\alpha x} \int e^{\alpha x} X dx + \frac{1}{\alpha-\beta} e^{-\beta x} \int e^{\beta x} X dx$$

vbi litterae α et β ita funt capiendae, vt fit

$$A + Bz + Czz = C(\alpha + z)(\beta + z)$$

Coroll. 1.

1144. Si bini hi factores fint aequales, feu $\beta = \alpha$, erit

$$X'' = e^{-\alpha x} \int dx \int e^{\alpha x} X dx = e^{-\alpha x} x \int e^{\alpha x} X dx - e^{-\alpha x} \int e^{\alpha x} X x dx$$

ideoque cafu $A + Bz + Czz = C(\alpha + z)^2$, aequationis noftrae integrale eft

$$Cy = e^{-\alpha x} (x \int e^{\alpha x} X dx - \int e^{\alpha x} X x dx).$$

Coroll. 2.

1145. Si bini factores fint imaginarii, quod euenit fi

$$A + Bz + Czz = C(ff + 2fz \operatorname{cof.} \theta + zz), \text{ erit}$$

$$\alpha = f(\operatorname{cof.} \theta + \sqrt{-1} . \operatorname{fin.} \theta) \text{ et } \beta = f(\operatorname{cof.} \theta - \sqrt{-1} . \operatorname{fin.} \theta)$$

hinc $e^{\alpha x} = e^{fx \operatorname{cof.} \theta} (\operatorname{cof.} fx \operatorname{fin.} \theta + \sqrt{-1} . \operatorname{fin.} fx \operatorname{fin.} \theta)$

et $e^{\beta x} = e^{fx \operatorname{cof.} \theta} (\operatorname{cof.} fx \operatorname{fin.} \theta - \sqrt{-1} . \operatorname{fin.} fx \operatorname{fin.} \theta)$

atque $\beta - \alpha = -2\sqrt{-1} . f \operatorname{fin.} \theta$.

Coroll. 3.

1146. Quo haec facilius substituere queamus, sit, breuitatis gratia

$$e^{f x \cos.\theta} = m; \quad \cos. f x \sin.\theta = p; \quad \text{et} \quad \sin. f x \sin.\theta = q,$$

vt fit

$$e^{\alpha x} = mp + mq\sqrt{-1}; \quad \text{et} \quad e^{\beta x} = mp - mq\sqrt{-1}.$$

Hinc fit

$$\int e^{\alpha x} X dx = \int mp X dx + \int mq X dx \sqrt{-1} \quad \text{et}$$

$$\int e^{\beta x} X dx = \int mp X dx - \int mq X dx \sqrt{-1}.$$

Tum vero est

$$e^{-\alpha x} = \frac{p - q\sqrt{-1}}{m}; \quad \text{et} \quad e^{-\beta x} = \frac{p + q\sqrt{-1}}{m}.$$

Coroll. 4.

1147. Ex his colligimus:

$$e^{-\alpha x}\int e^{\alpha x} X dx = \frac{p}{m}\int mp X dx - \frac{q\sqrt{-1}}{m}\int mp X dx + \frac{p\sqrt{-1}}{m}\int mq X dx + \frac{q}{m}\int mq X dx$$

et sumto $\sqrt{-1}$ negatiuo prodit $e^{-\beta x}\int e^{\beta x} X dx$, quae forma inde subtracta relinquit

$$-\frac{2q\sqrt{-1}}{m}\int mp X dx + \frac{2p\sqrt{-1}}{m}\int mq X dx$$

hocque residuum diuidi debet per

$$\beta - \alpha = -2\sqrt{-1}.f\sin.\theta.$$

Vnde integrale colligitur:

$$Cy = \frac{q}{m f \sin.\theta}\int mp X dx - \frac{p}{m f \sin.\theta}\int mq X dx.$$

Coroll. 4.

Coroll. 5.

1148. Restituantur pro m, p, q valores assumti atque aequationis nostrae, si fuerit

$$A + Bz + Czz = C(ff + 2fz\cos.\theta + zz)$$

integrale erit

$$Cy = e^{-fx\cos.\theta}\left(\frac{\sin.fx\sin.\theta}{\sin.\theta}\int e^{fx\cos.\theta}Xdx\cos.fx\sin.\theta - \frac{\cos.fx\sin.\theta}{\sin.\theta}\int e^{fx\cos.\theta}Xdx\sin.fx\sin.\theta\right)$$

quae ergo expressio aequiualet illi, si α et β valores imaginarios obtineant.

Problema 149.

1149. Proposita aequatione differentiali tertii gradus:

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3}$$

per triplicem integrationem eius integrale completum inuenire.

Solutio.

Posito multiplicatore $e^{\lambda x}dx$, debet esse

$$A - B\lambda + C\lambda^2 - D\lambda^3 = 0,$$

tum sumatur:

$$A' = B - C\lambda + D\lambda^2;\ B' = C - D\lambda \text{ et } C' = D$$

positoque

$$e^{-\lambda x}\int e^{\lambda x}Xdx = X',$$

aequatio semel integrata praebet

$$X' = A'y + B'\frac{dy}{dx} + C'\frac{ddy}{dx^2}.$$

 Huius

Huius porro multiplicator ſtatuatur $e^{\mu x}dx$, vt ſit

$$A' - B'\mu + C'\mu^2 = 0$$

ſumaturque

$$A'' = B' - C'\mu \quad \text{et} \quad B'' = C',$$

et poſito:

$$e^{-\mu x}\int e^{\mu x}X'dx = X''$$

aequatio ſecunda integralis eſt

$$X'' = A''y + B''\frac{dy}{dx}$$

cuius multiplicator erit $e^{\nu x}dx$ ſumendo $A'' - B''\nu = 0$ at poſito $A''' = B''$, erit aequatio integralis tertia

$$e^{-\nu x}\int e^{\nu x}X''dx = A'''y = Dy,$$

quaeri ergo oportet quantitates λ, μ, ν. Eſt vero primo

$$A - B\lambda + C\lambda^2 - D\lambda^3 = 0 \quad \text{tum}$$

$$B - C(\lambda + \mu) + D(\lambda\lambda + \lambda\mu + \mu\mu) = 0$$

et ob

$$A'' = C - D(\lambda + \mu) \quad \text{et} \quad B'' = D$$

erit tertio

$$C - D(\lambda + \mu + \nu) = 0$$

ex qua poſtrema aequalitate patet $\lambda + \mu + \nu$ aequari ſummae radicum aequationis primae; cuius λ eſt vna radix. Quod autem μ et ν ſint reliquae radices, hoc modo oſtenditur. Conſideretur aequatio:

$$A + Bz + Cz^2 + Dz^3 = 0$$

cuius

cuius si vna radix sit $z = -\lambda$, seu $\lambda + z$ vnus factor, diuidatur per eum aequatio, ac prodibit:

$$Dz^2 + (C - D\lambda)z + B - C\lambda + D\lambda\lambda = 0$$

quae est ipsa aequatio secunda $C'zz + B'z + A' = 0$, cuius radices sunt $z = -\mu$ et $z = -\nu$, seu factores $(\mu + z)(\nu + z)$ vti in problemate praecedente ostendimus. Quare si formulae

$$A + Bz + Cz^2 + Dz^3$$

factores sint

$$D(\alpha + z)(\beta + z)(\gamma + z)$$

pro integrali vltimo inueniendo ponatur:

$$e^{-\alpha x}\int e^{\alpha x} X\,dx = X'; \quad e^{-\beta x}\int e^{\beta x} X'\,dx = X'' \text{ et } e^{-\gamma x}\int e^{\gamma x} X''\,dx = X'''$$

eritque $Dy = X'''$. Verum per reductionem integralium est, vti supra vidimus:

$$X'' = \frac{1}{\beta - \alpha} e^{-\alpha x}\int e^{\alpha x} X\,dx + \frac{1}{\alpha - \beta} e^{-\beta x}\int e^{\beta x} X\,dx$$

hincque porro:

$$\int e^{\gamma x} X''\,dx = \frac{1}{(\beta - \alpha)(\gamma - \alpha)} e^{(\gamma - \alpha)x}\int e^{\alpha x} X\,dx - \frac{1}{(\beta - \alpha)(\gamma - \alpha)}\int e^{\gamma x} X\,dx$$
$$+ \frac{1}{(\alpha - \beta)(\gamma - \beta)} e^{(\gamma - \beta)x}\int e^{\beta x} X\,dx - \frac{1}{(\alpha - \beta)(\gamma - \beta)}\int e^{\gamma x} X\,dx$$

vbi bini postremi termini contrahuntur in

$$\frac{1}{(\alpha - \gamma)(\beta - \gamma)}\int e^{\gamma x} X\,dx.$$

Quamobrem integrale quaesitum est

$$Dy = \frac{e^{-\alpha x}\int e^{\alpha x} X\,dx}{(\beta - \alpha)(\gamma - \alpha)} + \frac{e^{-\beta x}\int e^{\beta x} X\,dx}{(\alpha - \beta)(\gamma - \beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X\,dx}{(\alpha - \gamma)(\beta - \gamma)}.$$

 Coroll. 1.

Coroll. 1.

1150. Si formulae $A + Bz + Cz^2 + Dz^3$ duo factores fuerint aequales, puta $\gamma = \beta$, erit

$$\int e^{\beta x} X^{II} dx = \frac{1}{(\beta-\alpha)^2} e^{(\beta-\alpha)x} \int e^{\alpha x} X dx - \frac{1}{(\beta-\alpha)^2} \int e^{\beta x} X dx$$
$$+ \frac{1}{\alpha-\beta} x \int e^{\beta x} X dx - \frac{1}{\alpha-\beta} \int e^{\beta x} X x dx$$

ideoque integrale hoc casu erit

$$Dy = \frac{e^{-\alpha x} \int e^{\alpha x} X dx - e^{-\beta x} \int e^{\beta x} X dx}{(\beta-\alpha)^2} + \frac{e^{-\beta x} x \int e^{\beta x} X dx - e^{-\beta x} \int e^{\beta x} X x dx}{\alpha-\beta}.$$

Coroll. 2.

1151. Si omnes tres factores sint aequales, seu $\alpha = \beta = \gamma$ erit

$$e^{\alpha x} X^{II} = \int dx \int e^{\alpha x} X dx = x \int e^{\alpha x} X dx - \int e^{\alpha x} X x dx \text{ et}$$
$$e^{\alpha x} X^{III} = \int e^{\alpha x} X^{II} dx = \int dx \int dx \int e^{\alpha x} X dx \text{ seu}$$
$$e^{\alpha x} X^{III} = \tfrac{1}{2} x x \int e^{\alpha x} X dx - x \int e^{\alpha x} X x dx + \tfrac{1}{2} \int e^{\alpha x} X x x dx$$

vnde integrale hoc casu erit

$$Dy = \tfrac{1}{2} e^{-\alpha x} (x x \int e^{\alpha x} X dx - 2 x \int e^{\alpha x} X x dx + \int e^{\alpha x} X x x dx)$$

seu $Dy = e^{-\alpha x} \int dx \int dx \int e^{\alpha x} X dx.$

Scholion.

1152. In genere etiam nulla integralium reductione adhibita integrale nostrae aequationis ita exprimi potest, vt sit

$$Dy = e^{-\gamma x} \int e^{(\gamma-\beta)x} dx \int e^{(\beta-\alpha)x} dx \int e^{\alpha x} X dx \text{ posito}$$
$$A + Bz + Cz^2 + Dz^3 = D(\alpha+z)(\beta+z)(\gamma+z)$$

vbi

vbi imprimis notatu dignum occurrit, quod ternas litteras α, β, γ quomodocunque inter se permutare licet, ita vt haec integralis expressio sex modis variari possit. In problemate etiam praecedente, vbi duo tantum factores occurrunt

$$C(\alpha+z)(\beta+z)=A+Bz+Cz^2,$$

aequationis

$$X=Ay+B\frac{dy}{dx}+C\frac{ddy}{dx^2}$$

integrale completum ita exhiberi potest:

$$Cy=e^{-\beta x}\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}X\,dx$$

ac permutatis litteris α et β etiam hoc modo:

$$Cy=e^{-\alpha x}\int e^{(\alpha-\beta)x}dx\int e^{\beta x}X\,dx.$$

Quarum formularum aequalitas si fuerit perspecta, id quod tentanti facile patebit, praecedentium quoque variationem declarat. Sit enim $e^{-\alpha x}\int e^{\alpha x}X\,dx=X'$, erit pro superiori formula:

$$Dy=e^{-\gamma x}\int e^{(\gamma-\beta)x}dx\int e^{\beta x}X'dx,$$

cui cum aequalis sit ista:

$$Dy=e^{-\beta x}\int e^{(\beta-\gamma)x}dx\int e^{\gamma x}X'dx$$

erit etiam pro X' valore restituto:

$$Dy=e^{-\beta x}\int e^{(\beta-\gamma)x}dx\int e^{(\gamma-\alpha)x}dx\int e^{\alpha x}X\,dx$$

quae a prima hoc tantum differt, quod litterae β et γ sunt permutatae. Quod autem etiam litterae β et γ cum α permutari queant, hoc modo difficilius ostenditur,

ditur, ex reductione autem in ſolutione adhibita, atque adeo ex ipſa ſolutionis indole per ſe eſt manifeſtum.

Problema 150.

1153. Propoſita aequatione differentiali quarti gradus:

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + E\frac{d^4y}{dx^4}$$

ſumto elemento dx conſtante, et denotante X functionem quamcunque ipſius x, eius integrale inueſtigare.

Solutio.

In ſubſidium vocetur formula algebraica ex aequatione propoſita facile formanda:

$$A + Bz + Czz + Dz^3 + Ez^4 = P$$

quae in factores ſuos ſimplices reſoluatur vt ſit

$$P = E(\alpha + z)(\beta + z)(\gamma + z)(\delta + z)$$

et multiplicator aequationem noſtram integrabilem reddens erit $e^{\lambda x}dx$ ſumendo λ aequali vni litterarum α, β, γ, δ; ſumatur ergo $\lambda = \alpha$, vt ſit multiplicator $e^{\alpha x}dx$, atque poſito $e^{-\alpha x}\int e^{\alpha x}X\,dx = X'$ aequatio ſemel integrata erit

$$X' = A'y + B'\frac{dy}{dx} + C'\frac{ddy}{dx^2} + D'\frac{d^3y}{dx^3}$$

vbi A', B', C', D' ita determinantur, vt ſit

$A' = \frac{A}{\alpha}$, $B' = \frac{-A + B\alpha}{\alpha^2}$, $C' = \frac{C\alpha^2 - B\alpha + A}{\alpha^3}$, $D' = \frac{D\alpha^3 - C\alpha^2 + B\alpha - A}{\alpha^4}$

ſeu $A' = \frac{A}{\alpha}$, $B' = \frac{B - A'}{\alpha}$, $C' = \frac{C - B'}{\alpha}$, $D' = \frac{D - C'}{\alpha}$

vel

vel etiam

$$A = A'\alpha\,;\; B = B'\alpha + A'\,;\; C = C'\alpha + B'\,;\; D = D'\alpha + C'.$$

Ex quibus determinationibus liquet, si ponatur

$$A' + B'z + C'z^2 + D'z^3 = Q$$

hanc formulam Q nasci ex formula P, si haec per $\alpha + z$ diuidatur, ita vt sit

$$Q = \frac{P}{\alpha + z} = E(\beta + z)(\gamma + z)(\delta + z).$$

Eodem ergo modo secundam integrationem instituamus ope multiplicatoris $e^{\beta x}dx$, et posito

$$e^{-\beta x}\int e^{\beta x}X'dx = X''$$

erit aequatio integralis

$$X'' = A''y + B''\frac{dy}{dx} + C''\frac{ddy}{dx^2}$$

coefficientibus A'', B'', C'' ita sumtis vt sit

$$A'' + B''z + C''z^2 = \frac{P}{(\alpha + z)(\beta + z)} = E(\gamma + z)(\delta + z).$$

Hinc porro ope multiplicatoris $e^{\gamma x}dx$ integrando, si ponamus $e^{-\gamma x}\int e^{\gamma x}X''dx = X'''$, inueniemus:

$$X''' = A'''y + B'''\frac{dy}{dx}$$

existente

$$A''' + B'''z = \frac{P}{(\alpha + z)(\beta + z)(\gamma + z)} = E(\delta + z).$$

Ac tandem ope multiplicatoris $e^{\delta x}dx$ posita forma

$$e^{-\delta x}\int e^{\delta x}X'''dx = X''''$$

integrale vltimum reperitur

$$X'''' = A''''y \text{ exiftente } A'''' = E.$$

Haec igitur omnia colligendo, integrale quaefitum erit

$$Ey = e^{-\delta x}\int e^{(\delta-\gamma)x}dx\int e^{(\gamma-\beta)x}dx\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}X\,dx$$

quae expreffio iam fine vllis ambagibus ex refolutione formae principalis

$$P = A + Bz + Cz^2 + Dz^3 + Ez^4$$

in factores fcilicet

$$P = E(\alpha + z)(\beta + z)(\gamma + z)(\delta + z),$$

confici poteft, vbi notandum quomodocunque ordo litterarum α, β, γ, δ permutetur, pro Ey femper eundem valorem prodire debere.

Coroll. 1.

1154. Cum fit $X' = e^{-\alpha x}\int e^{\alpha x}X\,dx$, erit vti iam vidimus

$$X'' = e^{-\beta x}\int e^{\beta x}X'dx = e^{-\beta x}\left(\frac{e^{(\beta-\alpha)x}}{\beta-\alpha}\int e^{\alpha x}Xdx - \frac{1}{\beta-\alpha}\int e^{\beta x}Xdx\right) \text{ feu}$$

$$X'' = \frac{e^{-\alpha x}\int e^{\alpha x}X\,dx}{\beta-\alpha} + \frac{e^{-\beta x}\int e^{\beta x}X\,dx}{\alpha-\beta}.$$

Coroll. 2.

Coroll. 2.

1155. Porro ob $X''' = e^{-\gamma x} \int e^{\gamma x} X'' dx$ erit similimodo reductionem instituendo:

$$X''' = \frac{e^{-\alpha x} \int e^{\alpha x} X dx}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\gamma x} \int e^{\gamma x} X dx}{(\beta-\alpha)(\alpha-\gamma)}$$
$$+ \frac{e^{-\beta x} \int e^{\beta x} X dx}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x} \int e^{\gamma x} X dx}{(\alpha-\beta)(\beta-\gamma)}$$

quae reducitur ad hanc formam:

$$X''' = \frac{e^{-\alpha x} \int e^{\alpha x} X dx}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\beta x} \int e^{\beta x} X dx}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x} \int e^{\gamma x} X dx}{(\alpha-\gamma)(\beta-\gamma)}$$

Coroll. 3.

1156. Hinc simili modo euoluitur valor X'''', vbi quidem sufficeret primum membrum eruisse, quippe ex quo ob permutabilitatem reliqua sponte formantur. Hoc modo integrale nostrae aequationis, reperietur hac forma expressum:

$$Ey = \frac{e^{-\alpha x} \int e^{\alpha x} X dx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} + \frac{e^{-\beta x} \int e^{\beta x} X dx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}$$
$$+ \frac{e^{-\gamma x} \int e^{\gamma x} X dx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} + \frac{e^{-\delta x} \int e^{\delta x} X dx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}.$$

Scholion.

1157. Si duae pluresue radices sint aequales vel imaginariae integralia inuenta transformationem po-

 stulant,

ſtulant, quam deinceps inueſtigabimus. Atque haec poſtrema quidem forma magis apta videtur, vnde transformationes repetantur. Ita pro factorum aequalitate ſi ſit $\delta=\gamma$, bina poſtrema membra tantum reductionem poſtulant, ad quam inueniendam ponatur $\delta=\gamma-\omega$, et penultimum membrum erit: $-\frac{e^{-\gamma x}\int e^{\gamma x}X\,dx}{\omega(\alpha-\gamma)(\beta-\gamma)}$ pro vltimo autem notandum eſt, eſſe:

$$\frac{1}{\alpha-\delta}=\frac{1}{\alpha-\gamma+\omega}=\frac{1}{\alpha-\gamma}-\frac{\omega}{(\alpha-\gamma)^2} \text{ et } \frac{1}{\beta-\delta}=\frac{1}{\beta-\gamma}-\frac{\omega}{(\beta-\gamma)^2},$$

hincque

$$\frac{1}{(\alpha-\delta)(\beta-\delta)}=\frac{1}{(\alpha-\gamma)(\beta-\gamma)}+\frac{\omega(2\gamma-\alpha-\beta)}{(\alpha-\gamma)^2(\beta-\gamma)^2},$$

vnde

$$\frac{1}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}=\frac{1}{\omega(\alpha-\gamma)(\beta-\gamma)}+\frac{2\gamma-\alpha-\beta}{(\alpha-\gamma)^2(\beta-\gamma)^2}.$$

Tum vero pro numeratore erit

$$e^{-\delta x}=e^{-\gamma x}(1+\omega x) \text{ et } e^{\delta x}=e^{\gamma x}(1-\omega x),$$

ideoque

$$e^{-\delta x}\int e^{\delta x}X\,dx=e^{-\gamma x}\int e^{\gamma x}X\,dx+e^{-\gamma x}\omega x\int e^{\gamma x}X\,dx$$
$$-\omega e^{-\gamma x}\int e^{\gamma x}Xx\,dx$$

atque hinc bina vltima membra ob terminos per ω diuiſos ſe deſtruentes, abeunt in hanc formam:

$$\frac{(2\gamma-\alpha-\beta)e^{-\gamma x}\int e^{\gamma x}X\,dx}{(\alpha-\gamma)^2(\beta-\gamma)^2}+\frac{e^{-\gamma x}x\int e^{\gamma x}X\,dx-e^{-\gamma x}\int e^{\gamma x}Xx\,dx}{(\alpha-\gamma)(\beta-\gamma)}$$

quae expreſſio etiam ex priori forma elicitur. Eodem modo problema in genere reſolui poteſt.

Proble-

Problema 151.

1158. Propofita aequatione differentiali cuiuscunque gradus:

$$X = Ay + B\frac{dy}{dx} + C\frac{ddv}{dx^2} + D\frac{d^3y}{dx^3} + \dots + N\frac{d^ny}{dx^n}$$

fumto elemento dx conftante et denotante X functionem quamcunque ipfius x, eius integrale inueftigare.

Solutio.

Formetur ex hac aequatione formula algebraica

$$A + Bz + Cz^2 + Dz^3 + \dots + Nz^n = P$$

quae in factores fimplices refoluatur, vt fit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)\dots(\nu + z)$$

quorum numerus eft n. Quodfi iam fimili modo per fingulas integrationes continuo progrediamur, tandem ad hanc aequationem integralem extremam perueniemus:

$$Ny = e^{-\nu x}\int e^{(\nu-\mu)x}dx\int e^{(\mu-\lambda)x}dx\int \dots \int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx$$

feu cum factores inter fe permutare liceat, erit etiam

$$Ny = e^{-\alpha x}\int e^{(\alpha-\beta)x}dx\int e^{(\beta-\gamma)x}dx\int \dots \int e^{(\mu-\nu)x}dx\int e^{\nu x}Xdx.$$

Haec vero expreffio per fimiles reductiones, quibus fupra fumus vfi, in fequentes partes refolui poteft, ad quas commodius repraefentandas fit breuitatis

 gratia:

gratia:

$$(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\ldots\ldots(\nu-\alpha)=\alpha'$$
$$(\alpha-\beta)(\gamma-\beta)(\delta-\beta)\ldots\ldots(\nu-\beta)=\beta'$$
$$(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)\ldots\ldots(\nu-\gamma)=\gamma'$$
$$\vdots$$
$$(\alpha-\nu)(\beta-\nu)(\gamma-\nu)\ldots\ldots(\mu-\nu)=\nu'$$

hincque erit:

$$Ny=\frac{1}{\alpha'}e^{-\alpha x}\int e^{\alpha x}Xdx+\frac{1}{\beta'}e^{-\beta x}\int e^{\beta x}Xdx+\frac{1}{\gamma'}e^{-\gamma x}\int e^{\gamma x}Xdx+\ldots+\frac{1}{\nu'}e^{-\nu x}\int e^{\nu x}Xdx.$$

Ne autem opus sit ad valores α', β', γ' etc. inveniendos tot factores in se inuicem multiplicare, cum sit

$$\frac{P}{N(\alpha+z)}=(\beta+z)(\gamma+z)(\delta+z)\ldots\ldots(\nu+z)$$

euidens est hanc formulam praebere valorem α', si in ea statuatur $z=-\alpha$; hoc autem casu fractionis $\frac{P}{N(\alpha+z)}$ tam numerator, quam denominator euanescit, ex quo eius valor erit $\frac{dP}{Ndz}$. Quare cum sit

$$P=A+Bz+Cz^2+Dz^3+\ldots+Nz^n$$

erit $\frac{dP}{dz}=B+2Cz+3Dz^2+\ldots+nNz^{n-1}$

quae expressio vocetur Q, vnde patet fore

$\alpha'=\frac{Q}{N}$ posito $z=-\alpha$
$\beta'=\frac{Q}{N}$ posito $z=-\beta$
$\gamma'=\frac{Q}{N}$ posito $z=-\gamma$.
etc.

Vel

Vel cum his valoribus substitutis aequatio integralis per N diuidi queat, sequentes valores colligantur:

$$B - 2C\alpha + 3D\alpha^2 - 4E\alpha^3 \ldots\ldots \pm nN\alpha^{n-1} = \mathfrak{A}$$
$$B - 2C\beta + 3D\beta^2 - 4E\beta^3 \ldots\ldots \pm nN\beta^{n-1} = \mathfrak{B}$$
$$B - 2C\gamma + 3D\gamma^2 - 4E\gamma^3 \ldots\ldots \pm nN\gamma^{n-1} = \mathfrak{C}$$
$$\vdots$$
$$B - 2C\nu + 3D\nu^2 - 4E\nu^3 \ldots\ldots \pm nN\nu^{n-1} = \mathfrak{N}$$

quibus constitutis erit integrale quaesitum:

$$y = \tfrac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X dx + \tfrac{1}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X dx + \tfrac{1}{\mathfrak{C}} e^{-\gamma x} \int e^{\gamma x} X dx \text{ etc.}$$

quoad omnes factores fuerint in computum ducti.

Coroll. 1.

1159. Cum sit

$$\alpha' = \tfrac{\mathfrak{A}}{N},\ \beta' = \tfrac{\mathfrak{B}}{N},\ \gamma' = \tfrac{\mathfrak{C}}{N} \text{ etc. erit}$$
$$\mathfrak{A} = N\alpha',\ \mathfrak{B} = N\beta',\ \mathfrak{C} = N\gamma' \text{ etc.}$$

Hinc ob

$$P = A + Bz + Cz^2 + Dz^3 \ldots\ldots Nz^n = N(\alpha+z)(\beta+z)(\gamma+z)\ldots\ldots(\nu+z)$$

erit

$$\mathfrak{A} = \tfrac{P}{\alpha+z} \text{ posito } z = -\alpha;$$
$$\mathfrak{B} = \tfrac{P}{\beta+z} \text{ posito } z = -\beta;$$
$$\mathfrak{C} = \tfrac{P}{\gamma+z} \text{ posito } z = -\gamma$$

et ita porro.

Coroll. 2.

Coroll. 2.

1160. Regula ergo huius aequationis propositae:

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \ldots\ldots + N\frac{d^ny}{dx^n}$$

integrale completum inueniendi ita se habet. Formetur inde formula algebraica haec

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n = P$$

cuius quaerantur omnes factores simplices, qui sint

$$\alpha + z,\ \beta + z,\ \gamma + z,\ \delta + z \text{ etc.}$$

quorum multitudo numero n est aequalis, tum pro singulis his factoribus sequentes quantitates constantes definiantur $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. vt sit

$\mathfrak{A} = \frac{P}{\alpha+z}$ posito $z = -\alpha$ seu $\mathfrak{A} = B - 2C\alpha + 3D\alpha^2 - 4E\alpha^3 \ldots \pm nN\alpha^{n-1}$

$\mathfrak{B} = \frac{P}{\beta+z}$ posito $z = -\beta$ seu $\mathfrak{B} = B - 2C\beta + 3D\beta^2 - 4E\beta^3 \ldots \pm nN\beta^{n-1}$

$\mathfrak{C} = \frac{P}{\gamma+z}$ posito $z = -\gamma$ seu $\mathfrak{C} = B - 2C\gamma + 3D\gamma^2 - 4E\gamma^3 \ldots \pm nN\gamma^{n-1}$

his omnibus inuentis erit integrale quaesitum:

$$y = \frac{1}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx + \frac{1}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx + \frac{1}{\mathfrak{C}}e^{-\gamma x}\int e^{\gamma x}Xdx + \text{etc.}$$

quae forma tot constat partibus, quot fuerint factores simplices in formula P.

Coroll. 3.

1161. Cum hoc modo integrale tot constet partibus, quoti ordinis est aequatio differentialis proposita,

posita, et vna quaeque pars per integrationem vnam inuehat constantem arbitrariam; manifestum est integrale ope huius regulae inuentum fore completum.

Scholion.

1162. Integratio ergo huiusmodi aequationum differentialium nulla amplius laborat difficultate, si modo formulae illius algebraicae P omnes factores simplices seu quod eodem redit, huius aequationis algebraicae

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n = 0$$

omnes radices numero n assign ri queant. Hic vero duplicis generis casus occurrunt, quibus haec integratio vehementer impeditur, quando scilicet vel duo pluresue eorum factorum simplicium inter se fiunt aequales, vel imaginarii, quo quidem posteriori casu hoc tantum incommodi accedit, quod partes quaepiam integralis inuenti imaginaria inuoluant, quae autem facta reductione se mutuo destruunt. Priori vero casu partes ex factoribus aequalibus oriundae adeo fiunt infinitae, sed ita diversis signis affectae, vt coniunctim nihilominus quantitatem finitam referant, cuius valorem nonnisi per plures ambages elicere licet, vbi probe notandum est, vtroque casu inuentionem reliquarum integralis partium, quae factoribus inaequalibus conueniunt, neutiquam hinc turbari. Methodum autem huic fini accommodatam in sequenti problemate explicabo.

Problema 152.

1163. Propoſita aequatione differentiali cuiuscunque gradus

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + E\frac{d^4y}{dx^4} + \dots + N\frac{d^ny}{dx^n}$$

ſi forma algebraica inde facta

$$P = A + Bz + Cz^2 + Dz^3 + Ez^4 + \dots + Nz^n$$

duos pluresue factores ſimplices inter ſe habeat aequales, partem integralis inde oriundam inueſtigare.

Solutio.

Sint primo duo factores $\alpha + z$ et $\beta + z$ inter ſe aequales ſeu $\beta = \alpha$, reliquus vero factor formae P ſit $= Q$ vt habeatur

$$P = (\alpha + z)(\beta + z)Q = (\alpha + z)^2 Q,$$

poſito autem $z = -\alpha$ abeat Q in $\mathfrak{C}$. Iam initio ſaltem litterae α et β vt diuerſae ſpectentur, excepta quantitate $\mathfrak{C}$ quae vtrinque ſit eadem, atque pro binis integralis partibus ex his binis factoribus oriundis habebimus:

$$\mathfrak{A} = (\beta - \alpha)\mathfrak{C} \text{ et } \mathfrak{B} = (\alpha - \beta)\mathfrak{C}.$$

Partes autem integralis inde oriundae littera v deſignentur vt ſit

$$(\beta - \alpha)\mathfrak{C}v = e^{-\alpha x}\int e^{\alpha x} X dx - e^{-\beta x}\int e^{\beta x} X dx$$

vnde

vnde differentiando colligimus

$$(\beta-\alpha)\mathfrak{C}dv=-\alpha e^{-\alpha x}dx\int e^{\alpha x}X\,dx+\beta e^{-\beta x}dx\int e^{\beta x}X\,dx$$

ad hanc addatur prior per βdx multiplicata, fietque

$$(\beta-\alpha)\mathfrak{C}dv+(\beta-\alpha)\mathfrak{C}\beta v dx=(\beta-\alpha)e^{-\alpha x}dx\int e^{\alpha x}X\,dx$$

quae per $\beta-\alpha$ diuisa, et per $e^{-\alpha x}$ multiplicata, ob $\beta=\alpha$ integrale praebet

$$\mathfrak{C}e^{\alpha x}v=\int dx\int e^{\alpha x}X\,dx.$$

Quocirca loco binarum partium ex factoribus aequalibus $\alpha+z$ et $\beta+z$ oriundarum scribi oportet hanc formulam

$$v=\tfrac{1}{\mathfrak{C}}e^{-\alpha x}\int dx\int e^{\alpha x}X\,dx$$

vbi $\mathfrak{C}$ oritur ex forma

$$\frac{P}{(\alpha+z)^3}\ \text{posito}\ z=-\alpha.$$

Ponamus iam formulam P tres habere factores simplices aequales, vt sit $\alpha+z=\beta+z=\gamma+z$, quosquidem initio vt diuersos spectemus.

Ponamus ergo $P=(\alpha+z)(\beta+z)(\gamma+z)Q$, abeatque Q in $\mathfrak{M}$ posito $z=-\alpha$, ac pro integralis partibus habebimus:

$$\mathfrak{A}=(\beta-\alpha)(\gamma-\alpha)\mathfrak{M};\ \mathfrak{B}=(\alpha-\beta)(\gamma-\beta)\mathfrak{M};$$
$$\mathfrak{C}=(\alpha-\gamma)(\beta-\gamma)\mathfrak{M}.$$

Hinc si summam trium integralis partium, quam quaerimus littera v denotemus erit

$$\mathfrak{M}v=\frac{e^{-\alpha x}\int e^{\alpha x}X\,dx}{(\beta-\alpha)(\gamma-\alpha)}+\frac{e^{-\beta x}\int e^{\beta x}X\,dx}{(\alpha-\beta)(\gamma-\beta)}+\frac{e^{-\gamma x}\int e^{\gamma x}X\,dx}{(\alpha-\gamma)(\beta-\gamma)}.$$

 Cum

Cum nunc fit

$$\frac{1}{(\beta-\alpha)(\gamma-\alpha)}+\frac{1}{(\alpha-\beta)(\gamma-\beta)}+\frac{1}{(\alpha-\gamma)(\beta-\gamma)}=0$$

erit differentiando

$$\frac{\mathfrak{M}dv}{dx}=\frac{-\alpha e^{-\alpha x}\int e^{\alpha x}X\,dx}{(\beta-\alpha)(\gamma-\alpha)}-\frac{\beta e^{-\beta x}\int e^{\beta x}X\,dx}{(\alpha-\beta)(\gamma-\beta)}-\frac{\gamma e^{-\gamma x}\int e^{\gamma x}X\,dx}{(\alpha-\gamma)(\beta-\gamma)}$$

ad quam fi prima per α multiplicata addatur fit

$$\mathfrak{M}\left(\frac{dv}{dx}+\alpha v\right)=\frac{e^{-\beta x}\int e^{\beta x}X\,dx}{\gamma-\beta}+\frac{e^{-\gamma x}\int e^{\gamma x}X\,dx}{\beta-\gamma}.$$

Haec aequatio denuo differentietur, vt prodeat

$$\mathfrak{M}\left(\frac{ddv}{dx^2}+\frac{\alpha dv}{dx}\right)=\frac{-\beta e^{-\beta x}\int e^{\beta x}X\,dx}{\gamma-\beta}-\frac{\gamma e^{-\gamma x}\int e^{\gamma x}X\,dx}{\beta-\gamma}$$

ad quam illa per β multiplicata fi addatur, oritur

$$\mathfrak{M}\left(\frac{ddv}{dx^2}+\frac{2\alpha dv}{dx}+\alpha\alpha v\right)=e^{-\gamma x}\int e^{\gamma x}X\,dx=e^{-\alpha x}\int e^{\alpha x}X\,dx$$

vnde iam omnia incommoda funt fublata. Multiplicetur nunc per $e^{\alpha x}dx$ et integratio dabit

$$\mathfrak{M}e^{\alpha x}\left(\frac{dv}{dx}+\alpha v\right)=\int dx\int e^{\alpha x}X\,dx$$

quae per dx multiplicata denuo fit integrabilis, proditque

$$\mathfrak{M}e^{\alpha x}v=\int dx\int dx\int e^{\alpha x}X\,dx.$$

Quocirca fi forma P factorem habeat cubicum $(\alpha+z)^3$ quaeratur quantitas $\mathfrak{M}$ vt fit

$$\mathfrak{M}=\frac{P}{(\alpha+z)^3}\text{ pofito } z=-\alpha$$

et

et integralis pars hinc orunda erit

$\frac{1}{\mathfrak{M}} e^{-\alpha x} \int dx \int dx \int e^{\alpha x} X\, dx.$

Simili modo si formula P quatuor habeat factores aequales vt sit $P = (\alpha + z)^4 Q$, capiatur $\mathfrak{M} = \frac{P}{(\alpha + z)^4}$ seu $\mathfrak{N} = Q$ posito $z = -\alpha$, et integralis pars inde nata erit:

$\frac{1}{\mathfrak{N}} e^{-\alpha x} \int dx \int dx \int dx \int e^{\alpha x} X\, dx.$

sicque etiam casus, quibus formula P adhuc plures habet factores aequales, facile resoluentur.

Nota. Tota haec solutio est vitiosa, propterea quod licet quantitates α, β, γ etc. quae ponuntur aequales, vt diuersae spectentur, tamen pro singulis membris quantitas $\mathfrak{M}$ eundem valorem retinere assumitur. Quodsi enim litterae α, β, γ etc. infinite parum a se inuicem discrepare concipiantur, etiam in valoribus littera $\mathfrak{M}$ indicatis differentiam infinite paruam agnoscere oportet, vnde cum singulae partes integralis fiant infinitae, iisque euolutis membra infinita se mutuo tollant, ex differentiis infinite paruis litterae $\mathfrak{M}$ partes quoque finitae emergunt. Correctionem horum errorum petere licet ex seq. Probl. 154, dum factores aequales in aequationem peculiarem coniiciuntur. Malui autem hunc correctionis laborem industriae lectorum relinquere, quam hoc opus a tali errore liberare, saepe enim plus prodest errores, in quos etiam exercitatis incidere contingit, conseruari, quo melius harum rerum studiosi addiscant quanta circumspectione cauendum sit, ne in ratiocinando hallucinemur.

 Coroll. 1.

Coroll. 1.

1164. Notatu hic omnino eſt dignum, quod hae formulae

$$dv + \alpha v dx;\quad ddv + 2\alpha dx dv + \alpha^2 v dx^2;$$
$$d^3v + 3\alpha dx ddv + 3\alpha^2 dx^2 dv + \alpha^3 v dx^3$$

et in genere haec

$$d^n v + \tfrac{n}{1}\alpha dx d^{n-1}v + \tfrac{n(n-1)}{1\cdot 2}\alpha^2 dx^2 d^{n-2}v + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\alpha^3 dx^3 d^{n-3}v \text{ etc.}$$

ſi ſemel per $e^{\alpha x}$ multiplicentur, ſucceſſiue toties integrationem admittant, quot vnitates continet index n ita vt poſtremum integrale ſit $e^{\alpha x}$.

Coroll. 2.

1165. Ratio autem huius phaenomeni inde eſt manifeſta, quod ſi formula $e^{\alpha x}v$ continuo differentietur ſumto elemento dx conſtante, formulae illae differentiales per $e^{\alpha x}$ multiplicatae prodeant, ita vt ſit:

$$d^n e^{\alpha x}v = e^{\alpha x}\left(d^n v + \tfrac{n}{1}\alpha dx d^{n-1}v + \tfrac{n(n-1)}{1\cdot 2}\alpha^2 dx^2 d^{n-2}v + \text{etc.}\right)$$

Coroll. 3.

1166. Aeque memoratu dignum eſt alterum phaenomenum, quod ſolutio iſta nobis offert, ſumtis ſcilicet numeris quibuscunque α, β, γ, δ etc. ſequentes aequalitates ſemper locum habere, vt ſit

$$\frac{1}{\alpha-\beta} + \frac{1}{\beta-\alpha} = 0$$
$$\frac{1}{(\alpha-\beta)(\alpha-\gamma)} + \frac{1}{(\beta-\alpha)(\beta-\gamma)} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)} = 0$$
$$\frac{1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{1}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{1}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} = 0$$

etc.

quotcunque numeri hoc modo capiantur.

Coroll. 4.

Coroll. 4.

1167. Si formula P in factores simplices resoluta ponatur

$$P = N(\alpha + z)(\beta + z)(\gamma + z) \ldots\ldots (\mu + z)(\nu + z)$$

expressio integralis prius inuenta (1158.) quae erat

$$Ny = e^{-\alpha x} \int e^{(\alpha - \beta)x} dx \int e^{(\beta - \gamma)x} dx \int e^{(\gamma - \delta)x} dx \ldots\ldots \int e^{(\mu - \nu)x} dx \int e^{\nu x} X dx$$

ob factores aequales nulla implicatur difficultate, forma autem posterior, qua integrale in partes ex singulis factoribus ortas distributum exhibetur, et quae ad vsum multo magis accommodata videtur, eo difficiliori egebat euolutione.

Scholion.

1168. Phaenomenum Coroll. 3. obseruatum eo maiorem attentionem meretur, quod etiam ad Arithmeticam vulgarem transferri potest, vbi vsu adeo insigni non cariturum videtur, praecipue quod eius demonstratio minime sit obuia sed ex profundioribus Analyseos penetralibus repeti debeat, ex quo haud alienum fore arbitror, si huic insigni Theoremati arithmetico hic locum concedam, idque eo magis, quod solutio problematis hic exposita sine demonstratione istius theorematis minime foret perfecta.

Theo-

Theorema Arithmeticum.

1169. Si habeantur numeri quotcunque a, b, c, d, etc. ex iisque dum a quolibet singuli reliqui subtrahantur formentur sequentia producta

$$(a-b)(a-c)(a-d)(a-e) \text{ etc. } = \mathfrak{A}$$
$$(b-a)(b-c)(b-d)(b-e) \text{ etc. } = \mathfrak{B}$$
$$(c-a)(c-b)(c-d)(c-e) \text{ etc. } = \mathfrak{C}$$
$$(d-a)(d-b)(d-c)(d-e) \text{ etc. } = \mathfrak{D}$$
$$\text{etc.}$$

semper habebitur:

$$\frac{1}{\mathfrak{A}} + \frac{1}{\mathfrak{B}} + \frac{1}{\mathfrak{C}} + \frac{1}{\mathfrak{D}} + \text{etc.} = 0.$$

Demonstratio.

Consideretur secundum principia in Introductione ad Analysin tradita haec fractio

$$\frac{Z}{(z-a)(z-b)(z-c)(z-d)\,\text{etc.}}$$

vbi Z denotet eiusmodi functionem rationalem integram ipsius z, in qua summa potestas ipsius z minor sit numero factorum denominatoris; haecque fractio resolui poterit in has fractiones simplices, quibus ea iunctim sumtis sit aequalis, scilicet:

$$\frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{z-c} + \frac{D}{z-d} + \text{etc.}$$

Ad quam resolutionem sumamus illum numeratorem $Z = z^n$ existente n numero integro minore quam deno-

denominator continet factores, atque hi numeratores ita definiuntur vt sit

$$A = \frac{a^n}{(a-b)(a-c)(a-d)\,\text{etc.}}$$

$$B = \frac{b^n}{(b-a)(b-c)(b-d)\,\text{etc.}}$$

$$C = \frac{c^n}{(c-a)(c-b)(c-d)\,\text{etc.}}$$

etc.

Cum igitur istae fractiones negatiue sumtae nempe

$$\frac{A}{a-z} + \frac{B}{b-z} + \frac{C}{c-z} + \frac{D}{d-z}\ \text{etc.}$$

ad fractionem propositam adiectae in nihilum abeant, si z sit numerorum propositorum a, b, c, d, etc. vltimus, quorum adeo multitudo maior est quam $n+1$, ponatur:

$$(a-b)(a-c)(a-d)\ldots\ldots(a-z) = \mathfrak{A}$$
$$(b-a)(b-c)(b-d)\ldots\ldots(b-z) = \mathfrak{B}$$
$$(c-a)(c-b)(c-d)\ldots\ldots(c-z) = \mathfrak{C}$$
$$(d-a)(d-b)(d-c)\ldots\ldots(d-z) = \mathfrak{D}$$

etc.

$$(z-a)(z-b)(z-c)\ldots\ldots(z-y) = \mathfrak{Z}$$

vt fractio proposita sit $\frac{z^n}{\mathfrak{Z}}$. Atque hinc perspicuum

eſt ſummam omnium harum fractionum eſſe:

$$\frac{a^n}{\mathfrak{A}}+\frac{b^n}{\mathfrak{B}}+\frac{c^n}{\mathfrak{C}}+\frac{d^n}{\mathfrak{D}}+\ldots\ldots+\frac{z^n}{\mathfrak{Z}}=0$$

dum ſit $n+1$ minor numero terminorum. Sumto ergo $n=0$ oritur caſus Theorematis.

Coroll. 1.

1170. Haec ſi transferantur ad numeros $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. ſupra (1160.) definitos, vbi aliquod leue diſcrimen in factorum conſtitutione probe eſt notandum, intelligetur eſſe

$$\frac{1}{\mathfrak{A}}+\frac{1}{\mathfrak{B}}+\frac{1}{\mathfrak{C}}+\frac{1}{\mathfrak{D}}+\text{etc.}=0$$

$$-\frac{\alpha}{\mathfrak{A}}-\frac{\beta}{\mathfrak{B}}-\frac{\gamma}{\mathfrak{C}}-\frac{\delta}{\mathfrak{D}}-\text{etc.}=0$$

$$+\frac{\alpha^2}{\mathfrak{A}}+\frac{\beta^2}{\mathfrak{B}}+\frac{\gamma^2}{\mathfrak{C}}+\frac{\delta^2}{\mathfrak{D}}+\text{etc.}=0$$

$$-\frac{\alpha^3}{\mathfrak{A}}-\frac{\beta^3}{\mathfrak{B}}-\frac{\gamma^3}{\mathfrak{C}}-\frac{\delta^3}{\mathfrak{D}}-\text{etc.}=0$$

etc.

donec perueniatur ad hanc formam

$$\pm\frac{\alpha^{n-1}}{\mathfrak{A}}\pm\frac{\beta^{n-1}}{\mathfrak{B}}\pm\frac{\gamma^{n-1}}{\mathfrak{C}}\pm\frac{\delta^{n-1}}{\mathfrak{D}}\pm\text{etc.}$$

cuius ſumma non amplius eſt euaneſcens, ſed aequalis fractioni $\frac{1}{\mathfrak{N}}$.

Coroll. 2.

Coroll. 2.

1171. Hoc etiam ex euolutione formae in Theoremate adhibitae colligere licet; etenim ſi ea ſtatuatur

$$\frac{z^{n-1}}{(z-a)(z-b)(z-c)\ldots(z-y)}$$

exiſtente omnium litterarum a, b, c etc. numero $=n$ quia hic numerator z^{n-1} tot habet dimenſiones, quot ſunt factores in denominatore, pars integra in hac fractione contenta eſt vnitas; quae etiam facta reſolutione conſeruatur, et in applicatione ad caſum memoratum abit in $\frac{1}{N}$.

Scholion.

1172. Poſt huius Theorematis demonſtrationem demum clare a poſteriori oſtendi poteſt, quemadmodum integrale ſupra (1160) exhibitum aequationi differentiali ibidem propoſitae ſatisfaciat. Notatis enim expreſſionibus §. 1170, cum ſupra inuenerimus integrale:

$$y=\frac{1}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}X\,dx+\frac{1}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}X\,dx+\text{etc.}$$

erit continuo differentiando:

$$\frac{dy}{dx}=-\frac{\alpha}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}X\,dx-\frac{\beta}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}X\,dx-\text{etc.}$$

$$\frac{ddy}{dx^2}=\frac{\alpha^2}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}X\,dx+\frac{\beta^2}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}X\,dx+\text{etc}$$

$$\frac{d^3y}{dx^3}=\frac{-\alpha^3}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}X\,dx-\frac{\beta^3}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}X\,dx-\text{etc.}$$

etc.

vsque ad

$$\frac{d^{n-1}y}{dx^{n-1}} = \pm \frac{\alpha^{n-1}}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X dx \pm \frac{\beta^{n-1}}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X dx \pm \text{etc.}$$

vnde sequens forma differentialis resultat:

$$\frac{d^n y}{dx^n} = \mp \frac{\alpha^n}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X dx \mp \frac{\beta^n}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X dx \mp \text{etc.}$$

$$\pm \left(\frac{\alpha^{n-1}}{\mathfrak{A}} + \frac{\beta^{n-1}}{\mathfrak{B}} + \frac{\gamma^{n-1}}{\mathfrak{C}} + \text{etc.} \right) X$$

quod postremum membrum abit in $\frac{1}{N} X$.

Si iam omnes hae formae singulatim multiplicentur per quantitates A, B, C, D . . . . . N, quoniam est

$$A - B\alpha + C\alpha^2 - D\alpha^3 + \ldots \mp N\alpha^n = 0$$

$$A - B\beta + C\beta^2 - D\beta^3 + \ldots \mp N\beta^n = 0$$

propterea quod $\alpha + z$, $\beta + z$, $\gamma + z$, etc. sunt factores formae

$$A + Bz + Cz^2 + Dz^3 + \ldots + Nz^n$$

manifesto obtinebimus:

$$Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \ldots + N\frac{d^ny}{dx^n} = X$$

quae est ipsa aequatio differentialis initio proposita.

Problema

Problema 153.

1173. Propofita aequatione differentiali cuiuscunque gradus:

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \ldots + N\frac{d^ny}{dx^n}$$

fi expreffio algebraica hinc formata

$$P = A + Bz + Cz^2 + Dz^3 \ldots\ldots + Nz^n$$

duos habeat factores fimplices imaginarios factore duplici $ff + 2fz\,\text{cof.}\,\theta + zz$ contentos, inueftigare partes integralis hinc oriundas.

Solutio.

Sint $\alpha + z$ et $\beta + z$, hi duo factores imaginarii, vt fit

$\alpha = f(\text{cof.}\,\theta + \sqrt{-1}.\,\text{fin.}\,\theta)$ et $\beta = f(\text{cof.}\,\theta - \sqrt{-1}.\,\text{fin.}\,\theta)$ ob

$$(\alpha + z)(\beta + z) = ff + 2fz\,\text{cof.}\,\theta + zz$$

ac ftatuatur $P = (ff + 2fz\,\text{cof.}\,\theta + zz)Q$, exiftente

$$Q = A' + B'z + C'z^2 + \ldots\ldots + N'z^{n-2}.$$

Cum igitur integralis partes ex binis illis factoribus fimplicibus imaginariis ortae fint

$$\tfrac{1}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}X\,dx + \tfrac{1}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}X\,dx = v$$

hos valores imaginarios ad realitatem perduci oportet. Erunt autem $\mathfrak{A}$ et $\mathfrak{B}$ quantitates imaginariae refultan-

sultantes ex forma

$$(f\cos.\theta \mp \sqrt{-1}.f\sin.\theta + z)Q$$

si loco z scribatur

$$-f\cos.\theta \mp \sqrt{-1}.f\sin.\theta.$$

At facta hac substitutione fit

$$Q = A' - B'f\cos.\theta + C'ff\cos.2\theta - D'f^3\cos.3\theta + \text{etc.}$$
$$\mp\sqrt{-1}.B'f\sin.\theta \pm \sqrt{-1}.C'ff\sin.2\theta \mp \sqrt{-1}.D'f^3\sin.3\theta \pm \text{etc.}$$

Ponamus breuitatis gratia:

$$A' - B'f\cos.\theta + C'ff\cos.2\theta - D'f^3\cos.3\theta + \text{etc} = \mathfrak{M} \text{ et}$$

$$-B'f\sin.\theta + C'ff\sin.2\theta - D'f^3\sin.3\theta + \text{etc.} = \mathfrak{N}$$

vt sit $Q = \mathfrak{M} \pm \mathfrak{N}\sqrt{-1}$ vbi signorum ambiguorum superius valet pro litteris α et $\mathfrak{A}$, inferius pro litteris β et $\mathfrak{B}$. Hinc ergo erit

$$\mathfrak{A} = -2\sqrt{-1}.f\sin.\theta(\mathfrak{M} + \mathfrak{N}\sqrt{-1}) \text{ et}$$

$$\mathfrak{B} = +2\sqrt{-1}.f\sin.\theta(\mathfrak{M} - \mathfrak{N}\sqrt{-1})$$

ideoque

$$2v\sqrt{-1}.f\sin.\theta = \frac{-e^{-\alpha x}\int e^{\alpha x}X\,dx}{\mathfrak{M} + \mathfrak{N}\sqrt{-1}} + \frac{e^{-\beta x}\int e^{\beta x}X\,dx}{\mathfrak{M} - \mathfrak{N}\sqrt{-1}}.$$

Est vero

$$e^{\alpha x} = e^{fx\cos.\theta}(\cos.(fx\sin.\theta) + \sqrt{-1}.\sin(fx\sin.\theta)) \text{ et}$$

$$e^{\beta x} = e^{fx\cos.\theta}(\cos.(fx\sin.\theta) - \sqrt{-1}.\sin.(fx\sin.\theta))$$

Sit

fit breuitatis gratia angulus $fx\sin.\theta = \Phi$ erit

$$2\sqrt{-1}.v(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})f\sin.\theta =$$

$$-(\mathfrak{M}-\mathfrak{N}\sqrt{-1})e^{-fx\cos.\theta}(\cos.\Phi-\sqrt{-1}.\sin.\Phi)\int e^{fx\cos.\theta}Xdx(\cos.\Phi+\sqrt{-1}.\sin.\Phi)$$

$$+(\mathfrak{M}+\mathfrak{N}\sqrt{-1})e^{-fx\cos.\theta}(\cos.\Phi+\sqrt{-1}.\sin.\Phi)\int e^{fx\cos.\theta}Xdx(\cos.\Phi-\sqrt{-1}.\sin.\Phi)$$

$$= e^{-fx\cos.\theta}\,2\sqrt{-1}.(\mathfrak{M}\sin.\Phi+\mathfrak{N}\cos.\Phi)\int e^{fx\cos.\theta}Xdx\cos.\Phi$$

$$-e^{-fx\cos.\theta}\,2\sqrt{-1}\,(\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi)\int e^{fx\cos.\theta}Xdx\sin.\Phi.$$

Quocirca habebimus integralis partem quaesitam

$$v = \frac{+e^{-fx\cos.\theta}(\mathfrak{M}\sin.\Phi+\mathfrak{N}\cos.\Phi)\int e^{fx\cos.\theta}Xdx\cos\Phi \atop -e^{-fx\cos.\theta}(\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi)\int e^{fx\cos.\theta}Xdx\sin.\Phi}{(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})f\sin.\theta}$$

existente $\Phi = fx\sin.\theta$.

Coroll. 1.

1174. Praecipuum igitur opus hic consistit in inuentione formulae imaginariae $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$, quae colligi debet ex quantitate Q, dum loco z scribitur valor imaginarius

$$-f(\cos.\theta+\sqrt{-1}.\sin.\theta),$$

vnde hoc commodi nascitur vt loco z^n scribi oporteat

$$(-f)^n(\cos.n\theta+\sqrt{-1}.\sin.n\theta).$$

Coroll. 2.

1175. Cum sit $Q = \frac{P}{ff+2fz\cos.\theta+zz}$ etiam ex hac forma per eandem substitutionem formula imaginaria $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$ inueniri potest, vbi autem

notan-

notandum est hac substitutione tam numeratorem P quam denominatorem euanescere. Ex quo manifestum est, valorem istius formulae rite obtineri ex hac fractione

$$\frac{dP}{2(f\cos.\theta+z)dz}=\frac{dP}{-2\sqrt{-1}.ff\sin.\theta dz}.$$

Coroll. 3.

1176. Quoniam igitur est

$$\frac{dP}{dz}=B+2Cz+3Dz^2+4Ez^3+\ldots nNz^{n-1}$$

si statuamus:

$$\mathfrak{P}=B-2Cf\cos.\theta+3Df^2\cos.2\theta-4Ef^3\cos.3\theta\ \ldots\ \pm nNf^{n-1}\cos.(n-1)\theta$$

$$\mathfrak{Q}=-2Cf\sin.\theta+3Df^2\sin.2\theta-4Ef^3\sin.2\theta\ \ldots\ \pm nNf^{n-1}\sin.(n-1)\theta$$

vt facta substitutione fiat

$$\frac{dP}{dz}=\mathfrak{P}+\sqrt{-1}.\mathfrak{Q},$$

habebimus

$$\mathfrak{M}+\sqrt{-1}.\mathfrak{N}=\frac{\mathfrak{P}+\sqrt{-1}.\mathfrak{Q}}{-2\sqrt{-1}.ff\sin.\theta}=\frac{-\mathfrak{Q}+\sqrt{-1}.\mathfrak{P}}{2ff\sin.\theta},$$

ideoque

$$\mathfrak{M}=\frac{-\mathfrak{Q}}{2ff\sin.\theta}\ \text{et}\ \mathfrak{N}=\frac{\mathfrak{P}}{2ff\sin.\theta}.$$

Coroll. 4.

1177. Immediate ergo ex quantitate P indeque deriuatis $\mathfrak{P}$ et $\mathfrak{Q}$, posito $fx\sin.\theta=\Phi$, integralis pars ex factore duplici $ff+2fz\cos.\theta+zz$ nata erit expressa:

$$v=\frac{2e^{-fx\cos.\theta}}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}\Big((\mathfrak{P}\cos.\Phi-\mathfrak{Q}\sin.\Phi)\textstyle\int e^{fx\cos.\theta}X\,dx\cos.\Phi$$
$$+(\mathfrak{P}\sin.\Phi+\mathfrak{Q}\cos.\Phi)\textstyle\int e^{fx\cos.\theta}X\,dx\sin.\Phi\Big).$$

Scholion.

Scholion.

1178. Quotcunque ergo forma

$$P = A + Bz + Cz^2 + Dz^3 + \text{etc.}$$

habuerit factores duplices, pro singulis ope horum praeceptorum partes integralis facile definiuntur, et quia hinc inuentio partium quae factoribus simplicibus conueniunt siue ii sint inaequales siue aequales non turbatur, omnibus partibus in vnam summam coniectis habebitur integrale completum aequationis differentialis propositae. Verum tamen haec praecepta non sufficiunt, si factorum duplicium bini pluresue inter se fuerint aequales; huiusmodi enim casus peculiarem exigunt euolutionem similem eius, qua pro casu duorum pluriumue factorum simplicium inter se aequalium sum vsus. Ne autem hanc tractationem nimis protraham, sufficiet casum pro duobus factoribus duplicibus inter se aequalibus euoluisse, cum inde methodus ad plures facile extendatur.

Problema 154.

1179. Proposita aequatione differentiali cuiuscunque gradus:

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \ldots + N\frac{d^ny}{dx^n}$$

si expressio algebraica inde formata

$$P = A + Bz + Cz^2 + Dz^3 + \ldots + Nz^n$$

 habeat

habeat factorem duplicem quadratum

$$(ff + 2fz\,\text{cof.}\,\theta + zz)^2$$

partem integralis ei conuenientem inuestigare.

Solutio.

Ponamus ergo $P = (ff + 2fz\,\text{cof.}\,\theta + zz)^2 Q$ fitque

$$Q = A' + B'z + C'zz + \ldots + N'z^{n-4}$$

ac primo imaginaria non curantes statuamus:

$\alpha = f(\text{cof.}\,\theta + \sqrt{-1}.\,\text{fin.}\,\theta)$ et $\beta = f(\text{cof.}\,\theta - \sqrt{-1}.\,\text{fin.}\,\theta)$ vt fit

$$P = (\alpha + z)^2 (\beta + z)^2 Q.$$

Iam ex iis quae fupra (1163.) de binis factoribus fimplicibus aequalibus docuimus, ponamus formam

$\frac{P}{(\alpha + z)^2} = (\beta + z)^2 Q$ pofito $z = -\alpha$ abire in $\mathfrak{A}$:

at hanc formam

$\frac{P}{(\beta + z)^2} = (\alpha + z)^2 Q$ pofito $z = -\beta$, in $\mathfrak{B}$,

quibus quantitatibus $\mathfrak{A}$ et $\mathfrak{B}$ inuentis ibi oftendi fore integralis partes hinc oriundas:

$$\tfrac{1}{\mathfrak{A}} e^{-\alpha x} \int dx \int e^{\alpha x} X\,dx + \tfrac{1}{\mathfrak{B}} e^{-\beta x} \int dx \int e^{\beta x} X\,dx = v$$

quas, cum iam imaginaria inuoluant, ad realitatem reduci oportet. Faciamus vt in problemate praecedente:

$$\mathfrak{M} = A' - B'f\,\text{cof.}\,\theta + C'f^2\,\text{cof.}\,2\theta - D'f^3\,\text{cof.}\,3\theta + \text{etc.}$$

$$\mathfrak{N} = \quad - B'f\,\text{fin.}\,\theta + C'f^2\,\text{fin.}\,2\theta - D'f^3\,\text{fin.}\,3\theta + \text{etc.}$$

vt

vt quantitas Q poſito

$$z = -\alpha = -f(\cos.\theta + \sqrt{-1}.\sin.\theta)$$

abeat in $\mathfrak{M} + \mathfrak{N}\sqrt{-1}$, at poſito

$$z = -\beta = -(f\cos.\theta - \sqrt{1}.\sin.\theta)$$

in $\mathfrak{M} - \mathfrak{N}\sqrt{-1}$. Cum iam ſit

$$(\beta - \alpha)^2 = (-2\sqrt{-1}.f\sin.\theta)^2 = -4ff\sin.\theta^2,$$

cui quoque $(\alpha - \beta)^2$ aequatur erit

$$\mathfrak{A} = -4ff\sin.\theta^2(\mathfrak{M} + \mathfrak{N}\sqrt{-1}) \text{ et } \mathfrak{B} = -4ff\sin.\theta^2(\mathfrak{M} - \mathfrak{N}\sqrt{-1})$$

ideoque

$$-4ff\sin.\theta^2(\mathfrak{M}\mathfrak{M} + \mathfrak{N}\mathfrak{N})v = (\mathfrak{M} - \mathfrak{N}\sqrt{-1})e^{-\alpha x}\int dx \int e^{\alpha x}X dx$$
$$+ (\mathfrak{M} + \mathfrak{N}\sqrt{-1})e^{-\beta x}\int dx \int e^{\beta x}X dx.$$

At poſito $fx \sin\theta = \Phi$ eſt vt vidimus:

$$e^{\alpha x} = e^{fx\cos.\theta}(\cos.\Phi + \sqrt{-1}.\sin.\Phi);\ e^{-\alpha x} = e^{-fx\cos.\theta}(\cos.\Phi - \sqrt{-1}.\sin.\Phi)$$
$$e^{\beta x} = e^{fx\cos.\theta}(\cos.\Phi - \sqrt{-1}.\sin.\Phi);\ e^{-\beta x} = e^{-fx\cos.\theta}(\cos.\Phi + \sqrt{-1}.\sin.\Phi)$$

Vnde illius aequationis alterum membrum abit in:

$$+e^{-fx\cos.\theta}(\mathfrak{M}\cos.\Phi - \mathfrak{N}\sin.\Phi - \mathfrak{N}\sqrt{-1}.\cos.\Phi - \mathfrak{M}\sqrt{-1}.\sin.\Phi)\int dx \int e^{fx\cos.\theta}X dx(\cos.\Phi + \sqrt{-1}.\sin.\Phi)$$
$$+e^{-fx\cos.\theta}(\mathfrak{M}\cos.\Phi - \mathfrak{N}\sin.\Phi + \mathfrak{N}\sqrt{-1}.\cos.\Phi + \mathfrak{M}\sqrt{-1}.\sin.\Phi)\int dx \int e^{fx\cos.\theta}X dx(\cos.\Phi - \sqrt{-1}.\sin.\Phi)$$

vbi partes imaginariae ſponte ſe deſtruunt, ita vt obtineatur:

$$v = \frac{-e^{-fx\cos.\theta}(\mathfrak{M}\cos.\Phi - \mathfrak{N}\sin.\Phi)\int dx \int e^{fx\cos.\theta}X dx \cos.\Phi - e^{-fx\cos.\theta}(\mathfrak{N}\cos.\Phi + \mathfrak{M}\sin.\Phi)\int dx \int e^{fx\cos.\theta}X dx \sin.\Phi}{2(\mathfrak{M}\mathfrak{M} + \mathfrak{N}\mathfrak{N})ff\sin.\theta^2}$$

 ſeu

seu hoc modo

$$v = \frac{-e^{-fx\cos.\theta}}{2(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})ff\sin.\theta^2}\left\{\begin{array}{l}+(\mathfrak{M}\cos\Phi-\mathfrak{N}\sin\Phi)\int dx\int e^{fx\cos.\theta}X\,dx\cos\Phi\\+(\mathfrak{M}\sin.\Phi+\mathfrak{N}\cos.\Phi)\int dx\int e^{fx\cos.\theta}X\,dx\sin.\Phi\end{array}\right.$$

quae expressio ab imaginariis penitus est liberata.

Nota. Etiam haec solutio insigni correctione indiget diligentiae lectorum relicta.

Coroll. 1.

1180. Quoniam formula imaginaria $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$ nascitur ex quantitate Q si loco z scribatur

$$-f(\cos.\theta+\sqrt{-1}.\sin.\theta)$$

eadem positione quoque reperietur ex forma

$$\frac{P}{(ff+2fz\cos.\theta+zz)^2}$$

verum hic tam numerator quam denominator prodit euanescens.

Coroll. 2.

1181. Orietur ergo quoque idem valor ex formula

$$\frac{dP}{4dz(f^3\cos.\theta+ffz(1+2\cos.\theta^2)+3fzz\cos.\theta+z^3)}$$

vbi cum idem incommodum denuo recurrat, orietur quoque ex hac formula

$$\frac{ddP}{4dz^2(ff(1+2\cos.\theta^2)+6fz\cos.\theta+3zz)}.$$

Coroll. 3.

Coroll. 3.

1182. Statuatur hic primo in denominatore $z = -f(\cos.\theta + \sqrt{-1}.\sin.\theta)$ fietque haec formula:

$$\frac{-ddP}{6ffdz^2 f\sin.\theta^2}.$$

Deinde cum sit

$$\frac{ddP}{2dz^2} = C + 3Dz + 6Ezz + \ldots \frac{n(n-1)}{1.2}Nz^{n-2}$$

ponamus breuitatis gratia:

$$\mathfrak{P} = C - 3Df\cos.\theta + 6Eff\cos.2\theta - \ldots \pm \frac{n(n-1)}{1.2}Nf^{n-2}\cos(n-2)\theta$$

$$\mathfrak{Q} = -3Df\sin.\theta + 6Eff\sin.2\theta - \ldots \pm \frac{n(n-1)}{1.2}Nf^{n-2}\sin.(n-2)\theta$$

vt sit facta substitutione

$$\frac{ddP}{2dz^2} = \mathfrak{P} + \mathfrak{Q}\sqrt{-1},$$

ideoque

$$\mathfrak{M} + \mathfrak{N}\sqrt{-1} = \frac{-\mathfrak{P} - \mathfrak{Q}\sqrt{-1}}{4fff\sin.\theta^2}$$

et consequenter

$$\mathfrak{M} = \frac{-\mathfrak{P}}{4fff\sin.\theta^2} \text{ et } \mathfrak{N} = \frac{-\mathfrak{Q}}{4fff\sin.\theta^2}.$$

Quos ergo valores in parte integralis inuenta substituere licet.

Coroll. 4.

1183. Facta autem substitutione, factor duplex quadratus $(ff + 2fz\cos.\theta + zz)^2$ hanc praebet integralis partem:

$$v = \frac{2e^{-fx\cos.\theta}}{\mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}} \left\{ \begin{array}{l} +(\mathfrak{P}\cos.\Phi - \mathfrak{Q}\sin.\Phi)\int dx\int e^{fx\cos.\theta}Xdx\cos.\Phi \\ +(\mathfrak{P}\sin.\Phi + \mathfrak{Q}\cos.\Phi)\int dx\int e^{fx\cos.\theta}Xdx\sin.\Phi \end{array} \right.$$

vbi Φ denotat angulum $fx\sin\theta$.

Scholion.

1184. Si hanc expreſſionem cum ea, quam problemate praecedente inuenimus, comparemus, vix actuali ſimili euolutione erit opus pro caſibus magis complicatis. Ita ſi quantitas

$$P = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n$$

factorem habeat duplicem cubicum

$$(ff + 2fz\cos.\theta + zz)^3$$

quantitates $\mathfrak{P}$ et $\mathfrak{Q}$ ita definiuntur, vt ſit

$$\mathfrak{P} = D - 4Ef\cos.\theta + 10Fff\cos.2\theta - 20Gf^3\cos.3\theta \ldots \pm \tfrac{n(n-1)(n-2)}{1.2.3} Nf^{n-3}\cos.(n-3)\theta$$

$$\mathfrak{Q} = -4Ef\sin\theta + 10Fff\sin.2\theta - 20Gf^3\sin.3\theta \ldots \pm \tfrac{n(n-1)(n-2)}{1.2.3} Nf^{n-3}\sin(n-3)\theta$$

quibus inuentis erit integralis pars hinc nata:

$$\frac{2e^{-fx\cos.\theta}}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}} \begin{cases} +(\mathfrak{P}\cos.\Phi - \mathfrak{Q}\sin.\Phi)\int dx\int dx\int e^{fx\cos.\theta} X dx \cos.\Phi \\ +(\mathfrak{P}\sin.\Phi + \mathfrak{Q}\cos.\Phi)\int dx\int dx\int e^{fx\cos.\theta} X dx \sin.\Phi \end{cases}$$

neque iam vlterior progreſſio vlli amplius difficultati eſt obnoxia. Quocirca aequationis hoc capite propoſitae reſolutionem ita concinne mihi equidem abſoluiſſe videor, vt nihil amplius deſiderari poſſit. Interim hoc argumentum maxime illuſtrabitur, ſi haec praecepta ad exempla particularia accommodabimus; cui inſtituto ſequens caput eſt deſtinatum. Ante autem inſignem proprietatem circa huiusmodi aequationes generales proponam, quae in Analyſi ingentem vſum habitura videtur.

Problema

Problema 155.

1185. Propofita aequatione differentiali cuiuscunque gradus:

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} \ldots\ldots + N\frac{d^{m+n}y}{dx^{m+n}}$$

fi formula algebraica inde nata:

$$P = A + Bz + Cz^2 + Dz^3 + \ldots + Nz^{m+n}$$

duobus factoribus conftet $P = QR$ vt fit

$$Q = \mathfrak{A} + \mathfrak{B}z + \mathfrak{C}z^2 + \ldots\ldots + \mathfrak{N}z^m \text{ et}$$

$$R = \mathfrak{a} + \mathfrak{b}z + \mathfrak{c}z^2 + \ldots\ldots + \mathfrak{n}z^n$$

integrationem illius aequationis ad integrationem binarum aequationum fimpliciorum reuocare.

Solutio.

Si formam integralem primo (1158.) perpendamus, haud difficulter inde colligimus, poftquam hanc aequationem integrauerimus:

$$X = \mathfrak{A}v + \mathfrak{B}\frac{dv}{dx} + \mathfrak{C}\frac{ddv}{dx^2} + \ldots + \mathfrak{N}\frac{d^m v}{dx^m}$$

indeque valorem ipfius v per x et X definiuerimus, valorem ipfius y pro aequatione propofita ex hac aequatione erutum iri:

$$v = \mathfrak{a}y + \mathfrak{b}\frac{dy}{dx} + \mathfrak{c}\frac{ddy}{dx^2} + \ldots + \mathfrak{n}\frac{d^n y}{dx^n}$$

cuius

cuius ratio adeo in promtu est posita; dum ex hac aequatione valores pro v eiusque differentialibus substituantur. Prodibit enim:

$$X = \mathfrak{A}ay + \mathfrak{A}\mathfrak{b}.\frac{dy}{dx} + \mathfrak{A}\mathfrak{c}.\frac{ddy}{dx^2} + \mathfrak{A}\mathfrak{d}.\frac{d^3y}{dx^3} \text{ etc.}$$
$$\quad + \mathfrak{B}a \quad + \mathfrak{B}\mathfrak{b} \quad + \mathfrak{B}\mathfrak{c}$$
$$\quad + \mathfrak{C}a \quad + \mathfrak{C}\mathfrak{b}$$
$$\quad + \mathfrak{D}a.$$

Cum autem per hypothesin sit $P = QR$, seriebus Q et R in se multiplicatis necesse est fieri

$$A = \mathfrak{A}a;\ B = \mathfrak{A}\mathfrak{b} + \mathfrak{B}a;\ C = \mathfrak{A}\mathfrak{c} + \mathfrak{B}\mathfrak{b} + \mathfrak{C}a \text{ etc.}$$

sicque haec postrema aequatio ad ipsam propositam reducitur

Coroll. 1.

1186. Si tantum ad factores simplices respiciamus, prioris aequationis integrale per huiusmodi terminos exprimitur

$$v = \Gamma e^{-\alpha x} \int e^{\alpha x} X dx \text{ etc.}$$

posterioris vero aequationis integrale per huiusmodi

$$y = \Delta e^{-\beta x} \int e^{\beta x} v dx \text{ etc.}$$

Coroll. 2.

1187. Quodsi iam in singulis terminis posterioris integralis substituamus singulos prioris, fiet

$$y = \Gamma \Delta e^{-\beta x} \int e^{(\beta - \alpha) x} dx \int e^{\alpha x} X dx$$

quae

quae forma ad hanc reducitur:

$$y = \frac{\Gamma\Delta}{\beta-\alpha}(e^{-\alpha x}\int e^{\alpha x} X dx - e^{-\beta x}\int e^{\beta x} X dx)$$

cuiusmodi termini per integrationem aequationis propositae immediate inueniuntur.

Coroll. 3.

1188. Si hic fuisset $\beta = \alpha$, sine vlla reductione statim prodiisset forma

$$y = \Gamma\Delta e^{-\alpha x}\int dx \int e^{\alpha x} X dx$$

supra pro casu duorum factorum simplicium aequalium inuenta. Interim cum totum negotium ad resolutionem in factores vel simplices vel duplices reales redeat, ipsa aequatio proposita modo ante exposita facillime expeditur.

CAPUT IV.

APPLICATIO METHODI INTEGRANDI IN CAPITE PRAECEDENTI TRADITAE AD EXEMPLA.

Problema 156.

1189.

Proposita hac aequatione differentiali

$$X = a^n y + \frac{d^n y}{dx^n}$$

eius integrale completum inuenire.

Solutio.

Hic ergo est $P = a^n + z^n$, vbi primo obseruetur, si n sit numerus impar factorem simplicem esse $a + z$, ex quo nascitur pars integralis

$$\frac{1}{\mathfrak{A}} e^{-ax} \int e^{ax} X\, dx,$$

existente $\mathfrak{A}$ valore ex forma $\frac{P}{a+z}$ emergente, si ponatur $z = -a$, qui ergo valor cum sit etiam $\frac{dP}{dz} = nz^{n-1}$, ob $n-1$ numerum parem erit $\mathfrak{A} = na^{n-1}$ ideoque haec integralis pars

$$= \frac{1}{na^{n-1}} e^{-ax} \int e^{ax} X\, dx.$$

Reliqui

Reliqui factores omnes in hac forma continentur: $aa-2az\cos.\theta+zz$, exiſtente $\theta=\frac{(2i+1)\pi}{n}$, vbi i denotat numerum integrum quemcunque et π angulum duobus rectis aequalem. Comparata hac forma cum Probl. 153. et Coroll. 1. fit $f=-a$ et ob $z=a(\cos.\theta+\sqrt{-1}.\sin.\theta)$ ex forma $\frac{dP}{dz}$ colligitur:

$$\mathfrak{P}=na^{n-1}\cos.(n-1)\theta \text{ et } \mathfrak{Q}=na^{n-1}\sin.(n-1)\theta$$

cum igitur ſit

$$\cos.n\theta=-1 \text{ et } \sin.n\theta=0 \text{ erit}$$

$$\mathfrak{P}=-na^{n-1}\cos.\theta \text{ et } \mathfrak{Q}=na^{n-1}\sin.\theta.$$

Quare poſito $fx\sin.\theta=-ax\sin.\theta=\Phi$, integralis pars ex quolibet factore duplici oriunda eſt:

$$\frac{2e^{ax\cos.\theta}}{na^{n-1}}\begin{cases}(-\cos.\theta\cos.\Phi-\sin.\theta\sin.\Phi)\int e^{-ax\cos.\theta}X\,dx\cos.\Phi\\(-\cos.\theta\sin.\Phi+\sin.\theta\cos.\Phi)\int e^{-ax\cos.\theta}X\,dx\sin.\Phi\end{cases}$$

ſeu

$$\frac{-2e^{ax\cos.\theta}}{na^{n-1}}(\cos.(\theta-\Phi)\int e^{-ax\cos.\theta}X\,dx\cos.\Phi-\sin.(\theta-\Phi)\int e^{-ax\cos.\theta}X\,dx\sin.\Phi)$$

et pro Φ valore reſtituto:

$$\frac{-2e^{ax\cos.\theta}}{na^{n-1}}(\cos.(\theta+ax\sin.\theta)\int e^{-ax\cos.\theta}X\,dx\cos.(ax\sin.\theta)$$
$$+\sin.(\theta+ax\sin.\theta)\int e^{-ax\cos.\theta}X\,dx\sin.(ax\sin.\theta)).$$

Iam pro θ ſucceſſiue ſubſtituantur anguli $\frac{\pi}{n}, \frac{3\pi}{n}, \frac{5\pi}{n}, \frac{7\pi}{n}$ quamdiu ipſo π ſunt minores, omnesque hae formae in vnam ſummam coniectae, quibus caſu quo n

 eſt

est numerus impar insuper addi oportet formam primo inuentam

$$\frac{1}{na^{n-1}} e^{-ax} \int e^{ax} X\, dx,$$

dabunt integrale quaesitum.

Coroll. 1.

1190. Casu quidem quo n est numerus impar, vltimus valor ipsius θ foret π, quem autem hic omitti iussimus, inde autem ob

$$ax \sin.\theta = 0 \text{ et } \cos.\theta = -1,$$

prodiret vltima pars integralis

$$\frac{2e^{-ax}}{na^{n-1}} \int e^{ax} X\, dx,$$

dupla eius quam capi conuenit; cuius ratio est, quod sumto $\theta = \pi$ formula $aa + 2az + zz$ non amplius ipsa est factor, sed eius radix quadrata $a + z$, ex quo hunc casum seorsim erui necesse erat.

Coroll. 2.

1191. Si est $X = 0$, formulae integrales abeunt in constantes arbitrarias et ex factore

$$aa - 2az \cos.\theta + zz$$

oritur haec pars integralis

$$\frac{-2e^{ax\cos.\theta}}{na^{n-1}} (A \cos.(\theta + ax \sin.\theta) + \mathfrak{A} \sin.(\theta + ax \sin.\theta))$$

quae

quae reducitur ad hanc formam

$$A\, e^{ax\cos.\theta}\cos.(\zeta + ax \sin.\theta)\ ,$$

denotante ζ angulum conftantem quemcunque, vti iam fupra inuenimus.

Problema 157.

1192. Propofita hac aequatione differentiali:

$$X = a^n y - \frac{d^n y}{dx^n}$$

eius integrale completum inuenire.

Solutio.

Forma algebraica hinc nata $P = a^n - z^n$ factorem femper habet $a-z$, vnde nafcitur pars integralis $\frac{1}{\mathfrak{A}} e^{ax} \int e^{-ax} X dx$ exiftente $\mathfrak{A} = \frac{P}{z-a}$ pofito $z=a$. Cum ergo fit quoque

$$\mathfrak{A} = \frac{dP}{dz} = -nz^{n-1}\ ,\ \text{erit}\ \mathfrak{A} = -na^{n-1}\ ,$$

ideoque haec pars integralis

$$= \frac{-1}{na^{n-1}} e^{ax} \int e^{-ax} X dx.$$

Deinde fi n fit numerus par, hincque $n-1$ impar, factor quoque erit $a+z$, qui praebet integralis partem

$$= \frac{1}{na^{n-1}}\ e^{-ax} \int e^{ax} X\, d\, x.$$

Reliqui factores omnes ipfius P funt duplicis formae

mae $aa - 2az\cos\theta + zz$ exiſtente angulo $\theta = \frac{2i\pi}{n}$, qua cum generali ſupra vſurpata $ff + 2fz\cos.\theta + zz$ comparata fit $f = -a$, et ex forma $\frac{dP}{dz} = -nz^{n-1}$ quaeri oportet formulam $\mathfrak{P} + \mathfrak{Q}\sqrt{-1}$, poſito $z = a(\cos.\theta + \sqrt{-1}.\sin.\theta)$, vnde colligitur:

$\mathfrak{P} = -na^{n-1}\cos.(n-1)\theta$ et $\mathfrak{Q} = -na^{n-1}\sin.(n-1)\theta$

ſeu ob $\cos.n\theta = 1$ et $\sin.n\theta = 0$, fit

$$\mathfrak{P} = -na^{n-1}\cos.\theta \text{ et } \mathfrak{Q} = +na^{n-1}\sin.\theta.$$

Poſito iam angulo $-ax\sin.\theta = \Phi$, ex §. 1177. oritur pars integralis:

$$\frac{2e^{ax\cos.\theta}}{na^{n-1}}\left\{\begin{array}{l}(-\cos.\theta\cos.\Phi - \sin.\theta\sin.\Phi)\int e^{-ax\cos.\theta}Xdx\cos.\Phi \\ +(-\cos.\theta\sin.\Phi + \sin.\theta\cos.\Phi)\int e^{-ax\cos.\theta}Xdx\sin.\Phi\end{array}\right.$$

quae vt ante reducitur ad hanc formam:

$$\frac{-2e^{ax\cos.\theta}}{na^{n-1}}(\cos.(\theta + ax\sin.\theta)\int e^{-ax\cos.\theta}Xdx\cos.(ax\sin.\theta)$$
$$+\sin.(\theta + ax\sin.\theta)\int e^{-ax\cos.\theta}Xdx\sin.(ax\sin.\theta)).$$

Hic iam pro θ ſucceſſiue ſcribantur anguli $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, $\frac{6\pi}{n}$ etc. quamdiu ſunt minores quam π, haeque partes omnes cum primum inuenta atque etiam altera, ſi n fuerit numerus par, in vnam ſummam collectae dabunt integrale quaeſitum ſeu valorem ipſius y.

Corollarium.

1193. Cum factor duplex generalis $aa - 2a\cos\theta + zz$ caſibus $\theta = 0$ et $\theta = \pi$ non praebeat ipſos factores ſimpli-

ſimplices reales $a-z$ et $a+z$ ſed eorum quadrata, haec ratio eſt, cur pars integralis inde eruta prodeat dupla eius, quam capi oportet.

Problema 158.

1194. Propoſita hac aequatione differentiali

$$X = y + \frac{dy}{dx} + \frac{ddy}{dx^2} + \frac{d^3y}{dx^3} \dots\dots + \frac{d^ny}{dx^n}$$

eius integrale completum inueſtigare.

Solutio.

Forma algebraica hinc nata eſt

$$P = 1 + z + z^2 + z^3 + z^4 \dots\dots + z^n$$

cuius omnes factores ſcrutari oportet. Cum igitur ſit $P = \frac{1 - z^{n+1}}{1-z}$, formae $1 - z^{n+1}$ factores capi conuenit excluſo $1-z$; vnde primo patet, ſi fuerit $n+1$ numerus par, factorem ſimplicem fore $1+z$, ex quo naſcitur pars integralis $\frac{1}{\mathfrak{A}} e^{-x} \int e^x X dx$ exiſtente $\mathfrak{A} = \frac{P}{1+z} = \frac{1-z^{n+1}}{1-zz}$ poſito $z = -1$. Erit ergo quoque $\mathfrak{A} = \frac{(n+1)z^n}{2z}$, ideoque $\mathfrak{A} = \frac{1}{2}(n+1)$, vt haec pars integralis ſit $\frac{2}{n+1} e^{-x} \int e^x X dx$.

Facto-

Factorum autem duplicium forma est $1-2z\cos\theta+zz$ sumto angulo $\theta=\frac{2i\pi}{n+1}$, ita vt pro §. 1176. sit $f=-1$. Consideretur forma

$$\frac{dP}{dz}=\frac{1-(n+1)z^{n}+nz^{n+1}}{(1-z)^{2}},$$

quae posito $z=\cos.\theta+\sqrt{-1}.\sin.\theta$ abire sumitur in $\mathfrak{P}+\mathfrak{Q}\sqrt{-1}$, sicque erit

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{1-(n+1)\cos.n\theta+n\cos.(n+1)\theta-(n+1)\sqrt{-1}.\sin.n\theta+n\sqrt{-1}.\sin.(n+1)\theta}{1-2\cos.\theta+\cos.2\theta-2\sqrt{-1}.\sin.\theta+\sqrt{-1}.\sin.2\theta}.$$

Cum vero sit

$$\sin.(n+1)\theta=0 \text{ et } \cos.(n+1)\theta=1, \text{ erit}$$

$$\sin.n\theta=-\sin.\theta \text{ et } \cos.n\theta=\cos.\theta, \text{ ideoque}$$

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{n+1-(n+1)\cos.\theta+(n+1)\sqrt{-1}.\sin.\theta}{-2\cos.\theta+2\cos.\theta^{2}-2\sqrt{-1}.\sin.\theta(1-\cos.\theta)} \text{ seu}$$

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{n+1}{2(1-\cos.\theta)}.\frac{1-\cos.\theta+\sqrt{-1}.\sin.\theta}{-\cos.\theta-\sqrt{-1}.\sin.\theta}$$

multiplicetur huius fractionis numerator et denominator per $-\cos.\theta+\sqrt{-1}.\sin.\theta$ et prodibit

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{-(n+1)(1+\cos.\theta-2\cos.\theta^{2}-\sqrt{-1}.\sin.\theta(1-2\cos.\theta))}{2(1-\cos.\theta)}$$

ita vt sit:

$$\mathfrak{P}=-\tfrac{1}{2}(n+1)(1+2\cos.\theta) \text{ et } \mathfrak{Q}=\tfrac{1}{2}(n+1)\frac{\sin.\theta(1-2\cos.\theta)}{1-\cos.\theta}$$

vnde fit

$$\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}=\frac{(n+1)^{2}}{2(1-\cos.\theta)}.$$

Tum vero posito angulo $-x\sin.\theta=\Phi$, colligitur

$$\mathfrak{P}\cos.\Phi-\mathfrak{Q}\sin.\Phi=\frac{-(n+1)(\cos.(\theta-\Phi)-\cos.(2\theta-\Phi))}{2(1-\cos.\theta)}$$

$$\mathfrak{P}\sin.\Phi+\mathfrak{Q}\cos.\Phi=\frac{+(n+1)(\sin.(\theta-\Phi)-\sin.(2\theta-\Phi))}{2(1-\cos.\theta)}$$

cum

cum autem sit

$\operatorname{cof.} a - \operatorname{cof.} b = 2 \operatorname{fin.} \frac{a+b}{2} \operatorname{fin.} \frac{b-a}{2}$ et $\operatorname{fin.} a - \operatorname{fin.} b = -2 \operatorname{cof.} \frac{a+b}{2} \operatorname{fin.} \frac{b-a}{2}$

fit hinc

$$\mathfrak{P} \operatorname{cof.} \Phi - \mathfrak{Q} \operatorname{fin.} \Phi = \frac{-(n+1) \operatorname{fin.} \frac{1}{2}(3\theta - 2\Phi)}{2 \operatorname{fin.} \frac{1}{2}\theta} \quad \text{et}$$

$$\mathfrak{P} \operatorname{fin.} \Phi + \mathfrak{Q} \operatorname{cof.} \Phi = \frac{-(n+1) \operatorname{cof.} \frac{1}{2}(3\theta - 2\Phi)}{2 \operatorname{fin.} \frac{1}{2}\theta}$$

ex quo integralis pars quaesita erit

$$\frac{-4}{n+1} e^{x \operatorname{cof.} \theta} \operatorname{fin.} \tfrac{1}{2}\theta \big(\operatorname{fin.} \tfrac{1}{2}(3\theta + 2x \operatorname{fin.} \theta) \textstyle\int e^{-x \operatorname{cof.} \theta} X dx \operatorname{cof.}(x \operatorname{fin.} \theta)$$
$$- \operatorname{cof.} \tfrac{1}{2}(3\theta + 2x \operatorname{fin.} \theta) \textstyle\int e^{-x \operatorname{cof.} \theta} X dx \operatorname{fin.}(x \operatorname{fin.} \theta) \big).$$

Pro θ ergo successiue substituantur anguli

$$\frac{2\pi}{n+1}; \quad \frac{4\pi}{n+1}; \quad \frac{6\pi}{n+1} \quad \text{etc.}$$

quamdiu sunt minores quam π haeque partes omnes in vnam summam colligantur, cui si $n+1$ sit numerus par addatur insuper $\frac{2}{n+1} e^{-x} \int e^{x} X dx$, sicque obtinebitur valor ipsius y.

Coroll. 1.

1195. Si aequatio proposita in infinitum progrediatur, vt sit n numerus infinitus, anguli θ priores omnes sunt infinite parui ideoque numero infiniti, quoad numerus par $2i$ ad $n+1$ rationem finitam habere incipiat, tum autem pro θ sequentur omnes anguli finiti in progressione arithmetica in-

crescentes, cuius differentia est $\frac{2\pi}{n+1}$, vsque ad π, quorum numerus itidem est infinitus.

Coroll. 2.

1196. Quamdiu angulus θ est infinite paruus, integralis pars ex eo oriunda hanc induit formam:

$$\frac{-\theta\theta e^{x}}{n+1}((3+2x)\int e^{-x}Xdx-2\int e^{-x}Xxdx)$$

quae cum per cubum infiniti sit diuisa, etiam multitudo infinita huiusmodi formularum pro euanescente est habenda.

Coroll. 3.

1197. Quodsi fuerit $X=0$, vt huius aequationis

$$0=y+\frac{dy}{dx}+\frac{ddy}{dx^2}+\frac{d^3y}{dx^3}+\ldots\ldots+\frac{d^ny}{dx^n}$$

integrale sit inuestigandum, erit eius pars quaecunque

$e^{x\cos.\theta}(A\,\text{fin.}\tfrac{1}{2}(3\theta+2x\,\text{fin.}\theta)+\mathfrak{A}\cos.\tfrac{1}{2}(3\theta+2x\,\text{fin.}\theta))$

seu simplicius

$A e^{x\cos.\theta}\cos.(\zeta+x\,\text{fin.}\theta)$.

Cum igitur si n fit numerus infinitus, pro θ angulus quicunque accipi queat, erit istius aequationis integrale particulare quodcunque:

$y=A e^{x\cos.\theta}\cos.(x\,\text{fin.}\theta+\zeta)$.

sumendo pro ζ etiam angulum quemcunque.

Scho-

Scholion.

1198. Num autem huius aequationis differentialis in infinitum excurrentis

$$X = y + \frac{dy}{dx} + \frac{ddy}{dx^2} + \frac{d^3y}{dx^3} + \frac{d^4y}{dx^4} + \text{etc.}$$

denotante X, functionem quamcunque ipsius x, integrale commodius exprimi possit, quam per partium illarum innumerabilium euanescentium summam, quaestio est altioris indaginis, neque adhuc ad hunc scopum Analyseos fines satis videntur promoti. Casibus quidem, quibus X est functio rationalis integra puta

$$X = a + bx + cx^2 + dx^3 + ex^4 + \text{etc.}$$

res nullam habet difficultatem, cum sumto

$$y = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.} + v$$

hi coefficientes α, β, γ etc. semper ita definiri queant, vt facta substitutione prodeat talis aequatio

$$o = v + \frac{dv}{dx} + \frac{ddv}{dx^2} + \frac{d^3v}{dx^3} + \text{etc.}$$

cui particulariter satisfacit valor

$$v = A\, e^{x \cos. \theta} \cos.(x \sin. \theta + \zeta)$$

sumtis pro ζ et θ angulis quibuscunque. Verum ex dato eiusmodi valore ipsius X inuenitur:

$$\alpha = a - b, \quad \beta = b - 2c, \quad \gamma = c - 3d, \quad \delta = d - 4e, \quad \varepsilon = e - 5f \text{ etc.}$$

Verum in genere cum fiat

$$\frac{dX}{dx} = \frac{dy}{dx} + \frac{ddy}{dx^2} + \frac{d^3y}{dx^3} + \text{etc.}$$

Mmm 2 euidens

euidens est semper posito $y = X - \frac{dX}{dx} + v$, aequationem illam transformari in hanc:

$$0 = v + \frac{dv}{dx} + \frac{ddv}{dx^2} + \frac{d^3v}{dx^3} + \text{etc.}$$

Corollarium.

1199. En ergo praeter expectationem integrationem completam huius aequationis differentialis in infinitum excurrentis:

$$X = y + \frac{dy}{dx} + \frac{ddy}{dx^2} + \frac{d^3y}{dx^3} + \frac{d^4y}{dx^4}\text{etc. in infinitum}$$

pro qua iam nouimus esse

$$y = X - \frac{dX}{dx} + Ae^{x\cos.\theta}\cos.(x\sin.\theta + \zeta)$$

quod postremum membrum ob angulos ζ et θ arbitrarios in infinitum multiplicari potest. Haecque forma maxime complicatae illi ex solutione oriundae aequiualere est censenda.

Problema 159.

1200. Proposita hac aequatione differentiali

$$X = y + \frac{ndy}{adx} + \frac{n(n-1)ddy}{1\cdot 2\cdot a^2dx^2} + \frac{n(n-1)(n-2)d^3y}{1\cdot 2\cdot 3\,a^3dx^3} + \text{etc.}$$

vbi quidem n sit numerus integer affirmatiuus, vt terminorum numerus sit finitus, eius integrale completum inuestigare.

Solutio.

Formula algebraica hinc consideranda fit

$$P = 1 + \frac{n}{1}\frac{z}{a} + \frac{n(n-1)}{1\cdot 2}\cdot\frac{z^2}{a^2} + \text{etc.} = (1 + \frac{z}{a})^n$$

quae

quae ergo meros habet factores simplices inter se aequales $z+a$. Cum igitur sit $\frac{P}{(a+z)^n}=\frac{1}{a^n}$ ex §. 1163. statim colligitur integrale quaesitum

$$y=a^n e^{-ax}\int dx\int dx\int dx\ \ldots\ldots \int e^{ax}X\,dx$$

quoad signorum integralium numerus aequetur exponenti n. Hanc autem formam sequenti modo in integralia simplicia resoluere licet, ope reductionis generalis qua esse nouimus

$$\int dx\int V\,dx=x\int V\,dx-\int Vx\,dx\,,$$

vnde fit

$$\int dx\int e^{ax}X\,dx=x\int e^{ax}X\,dx-\int e^{ax}Xx\,dx$$

$$\int dx\int dx\int e^{ax}X\,dx=\tfrac{1}{2}x^2\int e^{ax}X\,dx-x\int e^{ax}Xx\,dx+\tfrac{1}{2}\int e^{ax}Xx^2\,dx$$

$$\int dx\int dx\int dx\int e^{ax}X\,dx=\frac{x^3\int e^{ax}X\,dx-3x^2\int e^{ax}Xx\,dx+3x\int e^{ax}Xx^2\,dx-\int e^{ax}Xx^3\,dx}{1.\ 2.\ 3}$$

etc.

Cum igitur signorum integralium numerus sit $=n$, concludimus fore

$$y=\frac{a^n e^{-ax}}{1.2\ldots\ldots(n-1)}\left(x^{n-1}\int e^{ax}X\,dx-\frac{(n-1)}{1}x^{n-2}\int e^{ax}Xx\,dx\right.$$
$$\left.+\frac{(n-1)(n-2)}{1\cdot 2}x^{n-3}\int e^{ax}Xx^2\,dx-\text{etc.}\right)$$

vbi cum singula integralia constantem arbitrariam implicent manifestum est, hoc integrale esse completum.

 Coroll. 1.

Coroll. 1.

1201. Si ergo esset $X = 0$ aequationis differentialis propositae integrale completum foret:

$$y = e^{-ax}(Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \text{etc.} \ldots + Mx + N)$$

vbi constantium arbitrariarum A, B, C etc. numerus vtique est $= n$.

Coroll. 2.

1202. Si numerus n fuerit infinitus, simulque quantitas a capiatur infinita vt sit $a = nc$, aequatio integranda in infinitum excurret, eritque

$$X = y + \frac{dy}{c\,dx} + \frac{ddy}{1 \cdot 2\, c^2\, dx^2} + \frac{d^3 y}{1 \cdot 2 \cdot 3\, c^3\, dx^3} + \text{etc.}$$

aequatio autem integralis ad hunc casum applicata nullam lucem foeneratur.

Coroll. 3.

1203. Quaecunque autem y functio fuerit ipsius x, constat si loco x scibatur $x + \frac{1}{c}$, eam abire in:

$$y + \frac{dy}{c\,dx} + \frac{ddy}{1 \cdot 2\, c^2\, dx^2} + \frac{d^3 y}{1 \cdot 2 \cdot 3\, c^3\, dx^3} + \text{etc.}$$

quae cum esse debeat $= X$, vicissim patet y aequari ei functioni ipsius x quae nascitur ex X, si ibi loco x scribatur $x - \frac{1}{c}$.

Scho-

Scholion 1.

1204. Quod quo facilius appareat obferuo fi propofita fuerit quaecunque eiusmodi aequatio

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \text{etc.}$$

femper fine vlla integratione integrale particulare per approximationem hoc modo inueniri poffe, ftatuatur

$$y = \alpha X + \beta\frac{dX}{dx} + \gamma\frac{ddX}{dx^2} + \delta\frac{d^3X}{dx^3} + \text{etc.}$$

factaque fubftitutione habebitur:

$$\begin{array}{llll} X = A\alpha X + & A\beta.\frac{dX}{dx} + & A\gamma.\frac{ddX}{dx^2} + & A\delta.\frac{d^3X}{dx^3} + \text{etc.} \\ & +B\alpha & +B\beta & +B\gamma \\ & & +C\alpha & +C\beta \\ & & & +D\alpha \end{array}$$

ficque coefficientes α, β, γ, δ etc. definiuntur, vt fit $\alpha = \frac{1}{A}$, reliqui vero

$$\beta = \frac{-B\alpha}{A} = \frac{-B}{A^2}$$

$$\gamma = \frac{-C\alpha - B\beta}{A} = \frac{-C}{A^2} + \frac{BB}{A^3}$$

$$\delta = \frac{-D\alpha - C\beta - B\gamma}{A} = \frac{-D}{A^2} + \frac{2BC}{A^3} - \frac{B^3}{A^4}$$

$$\varepsilon = \frac{-E\alpha - D\beta - C\gamma - D\delta}{A} = \frac{-E}{A^2} + \frac{2BD + C^2}{A^3} - \frac{3BBC}{A^4} + \frac{B^4}{A^5}$$

etc.

quae fi accommodentur ad cafum problematis, fiet

$$y = X - \frac{n\,dX}{1a\,dx} + \frac{n(n+1)ddX}{1.2\,a^2\,dx^2} - \frac{n(n+1)(n+2)d^3X}{1.2.3\,a^3\,dx^3} + \text{etc.}$$

Hinc

Hinc casu quo $n = \infty$ et $a = nc$ colligitur:

$$y = X - \frac{dX}{1 c dx} + \frac{ddX}{1.2 c^2 dx^2} - \frac{d^3 X}{1.2.3 c^3 dx^3} + \text{etc.}$$

quae expressio etsi in infinitum excurrens manifesto definit eam ipsius x functionem quae nascitur ex X si loco x scribatur $x - \frac{1}{c}$. Quodsi iam hanc nouam functionem signo X' indicemus ponamusque $y = X' + v$ aequatio Coroll. 2. abit in hanc:

$$0 = v + \frac{dv}{dx} + \frac{ddv}{1.2 dx^2} + \frac{d^3 v}{1.2.3 dx^3} + \text{etc.}$$

cuius integrale particulare quodcunque est $v = Ae^{-ncx} x^m$ existente n numero infinito, et m numero integro positiuo.

Scholion 2.

1205. Haec me deducunt ad sequentem speculationem circa serierum summationem. Sit nempe series quaecunque

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & & x \\ A, & B, & C, & D & \ldots\ldots & T \end{array}$$

cuius terminus indici x respondens sit T functio quaecunque ipsius x. Statuatur summa omnium horum terminorum

$$A + B + C + D + \ldots\ldots + T = y,$$

ac perspicuum est y fore eiusmodi functionem ipsius x, vt si in ea loco x scribatur $x - 1$, proditura sit eadem illa summa y termino vltimo T mulctata, scilicet

ſcilicet $y-T$. At loco x ſcribendo $x-1$, functio y abit in:

$$y-\frac{dy}{dx}+\frac{ddy}{1\cdot 2\,dx^2}-\frac{d^3y}{1\cdot 2\cdot 3\,dx^3}+\text{etc.}$$

vnde oritur haec aequatio:

$$T=\frac{dy}{dx}-\frac{ddy}{1\cdot 2\,dx^2}+\frac{d^3y}{1\cdot 2\cdot 3\,dx^3}-\frac{d^4y}{1\cdot 2\cdot 3\cdot 4\,dx^4}+\text{etc.}$$

quae ſemel integrata poſito $\int T\,dx=X$ fit

$$X=y-\frac{dy}{1\cdot 2\,dx}+\frac{ddy}{1\cdot 2\cdot 3\,dx^2}-\frac{d^3y}{1\cdot 2\cdot 3\cdot 4\,dx^3}+\text{etc.}$$

quam quomodo integrari conueniat videamus, dum eam aliquanto generaliorem reddemus.

Problema 160.

1206. Propoſita hac aequatione differentiali:

$$X=\frac{ny}{a}-\frac{n(n-1)\,dy}{1\cdot 2\,a^2\,dx}+\frac{n(n-1)(n-2)\,ddy}{1\cdot 2\cdot 3\,a^3\,dx^2}-\text{etc.}$$

eius integrale completum inueſtigare.

Solutio.

Formetur inde haec quantitas algebraica:

$$P=\frac{n}{a}-\frac{n(n-1)z}{1\cdot 2\,a^2}+\frac{n(n-1)(n-2)zz}{1\cdot 2\cdot 3\,a^3}-\text{etc.}=\frac{1-\left(1-\frac{z}{a}\right)^n}{z}$$

ſeu $P=\dfrac{a^n-(a-z)^n}{a^n z}$, cuius factor duplex quicunque hanc habebit formam

$$aa-2a(a-z)\cos.2\zeta+(a-z)^2$$

 exiſtente

exiſtente angulo $2\zeta = \frac{2i\pi}{n}$. Abit autem haec forma in

$$2aa(1-\text{coſ.}\,2\zeta) - 2az(1-\text{coſ.}\,2\zeta) + zz = 4aa\,\text{ſin.}\,\zeta^2 - 4az\,\text{ſin.}\,\zeta^2 + zz$$

quae cum generali $ff + 2fz\,\text{coſ.}\,\theta + zz$ comparata dat:

$$f = 2a\,\text{ſin.}\,\zeta \text{ et } \text{coſ.}\,\theta = -\,\text{ſin.}\,\zeta$$

vnde

$$\theta = 90^\circ + \zeta \text{ et } \text{ſin.}\,\theta = \text{coſ.}\,\zeta$$

exiſtente $\zeta = \frac{i\pi}{n}$. Iam ad partem integralis hinc ortam inueniendam conſideretur forma:

$$\frac{dP}{dz} = \frac{-a^n + (a + (n-1)z)(a-z)^{n-1}}{a^n zz},$$

in qua poſito

$$z = -f(\text{coſ.}\,\theta + \sqrt{-1}.\,\text{ſin.}\,\theta) \text{ ſeu}$$

$$z = 2a\,\text{ſin.}\,\zeta(\text{ſin.}\,\zeta - \sqrt{-1}.\,\text{coſ.}\,\zeta) = a(1 - \text{coſ.}\,2\zeta - \sqrt{-1}.\,\text{ſin.}\,2\zeta)$$

vt ſit

$$a - z = a(\text{coſ.}\,2\zeta + \sqrt{-1}.\,\text{ſin.}\,2\zeta)$$

prodit

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{-1 + (n-(n-1)(\text{coſ.}\,2\zeta + \sqrt{-1}.\,\text{ſin.}\,2\zeta))(\text{coſ.}\,2(n-1)\zeta + \sqrt{-1}.\,\text{ſin.}\,2(n-1)\zeta)}{-4aa\,\text{ſin.}\,\zeta^2(\text{coſ.}\,2\zeta + \sqrt{-1}.\,\text{ſin.}\,2\zeta)}.$$

Cum autem ſit

$$\text{coſ.}\,2n\zeta = 1 \text{ et } \text{ſin.}\,2n\zeta = 0, \text{ erit}$$

$$\text{coſ.}\,2(n-1)\zeta = \text{coſ.}\,2\zeta \text{ et } \text{ſin.}\,2(n-1)\zeta = -\,\text{ſin.}\,2\zeta$$

ideoque

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{-n + n(\text{coſ.}\,2\zeta - \sqrt{-1}.\,\text{ſin.}\,2\zeta)}{-4aa\,\text{ſin.}\,\zeta^2(\text{coſ.}\,2\zeta + \sqrt{-1}.\,\text{ſin.}\,2\zeta)}$$

quae

quae reducitur ad hanc formam

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{n}{4aa\,\mathrm{fin.}\,\zeta^2}(\mathrm{cof.}\,2\zeta-\sqrt{-1}.\,\mathrm{fi}\,2\zeta-\mathrm{cf.}\,4\zeta+\sqrt{-1}.\,\mathrm{fi.}\,4\zeta)$$

vnde concluditur

$$\mathfrak{P}=\frac{n}{4aa\,\mathrm{fin.}\,\zeta^2}(\mathrm{cof.}\,2\zeta-\mathrm{cof.}\,4\zeta)=\frac{n}{2aa\,\mathrm{fin.}\,\zeta}\,\mathrm{fin.}\,3\zeta$$

$$\mathfrak{Q}=\frac{-n}{4aa\,\mathrm{fin.}\,\zeta^2}(\mathrm{fin.}\,2\zeta-\mathrm{fin.}\,4\zeta)=\frac{n}{2aa\,\mathrm{fin.}\,\zeta}\,\mathrm{cof.}\,3\zeta$$

ſicque eſt

$$\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}=\frac{nn}{4a^4\,\mathrm{fin.}\,\zeta^2}$$

et poſito

$$\Phi=2ax\,\mathrm{fin.}\,\zeta\,\mathrm{cof.}\,\zeta=ax\,\mathrm{fin.}\,2\zeta \text{ fiet}$$

$$\mathfrak{P}\,\mathrm{cof.}\,\Phi-\mathfrak{Q}\,\mathrm{fin.}\,\Phi=\frac{n}{2aa\,\mathrm{fin.}\,\zeta}\,\mathrm{fin.}(3\zeta-\Phi) \text{ et}$$

$$\mathfrak{P}\,\mathrm{fin.}\,\Phi+\mathfrak{Q}\,\mathrm{cof.}\,\Phi=\frac{n}{2aa\,\mathrm{fin.}\,\zeta}\,\mathrm{cof.}(3\zeta-\Phi).$$

Quocirca integralis pars hinc oriunda erit

$$\frac{4aa\,\mathrm{fin.}\,\zeta}{n}e^{2ax\,\mathrm{fin.}\,\zeta^2}\Big(\mathrm{fin.}(3\zeta-\Phi)\int e^{-2ax\,\mathrm{fin.}\,\zeta^2}X\,dx\,\mathrm{cof.}\,\Phi$$
$$+\mathrm{cof.}(3\zeta-\Phi)\int e^{-2ax\,\mathrm{fin.}\,\zeta^2}X\,dx\,\mathrm{fin.}\,\Phi\Big)$$

vbi pro ζ ſucceſſiue ſcribi debent hi anguli

$$\frac{\pi}{n},\ \frac{2\pi}{n},\ \frac{3\pi}{n},\ \frac{4\pi}{n} \text{ etc.}$$

quamdiu ſunt angulo recto minores, at ſi n ſit numerus par ad has partes inſuper addi oportet

$$-\frac{2aa}{n}e^{2ax}\int e^{-2ax}X\,dx,$$

ſicque colligetur verus valor ipſius y.

Coroll. 1.

1207. Si eſt $X=0$, pars integralis ex quolibet angulo $\zeta=\frac{i\pi}{n}$ nata induit hanc formam

$$e^{2ax\,\text{fin.}\,\zeta^2}(A\,\text{fin.}(3\zeta-ax\,\text{fin.}\,2\zeta)+B\,\text{cof.}(3\zeta-ax\,\text{fin.}\,2\zeta))$$

ſeu hanc

$$Ae^{2ax\,\text{fin.}\,\zeta^2}\,\text{fin.}(\alpha+ax\,\text{fin.}\,2\zeta)$$

denotante α angulum quemcunque conſtantem.

Coroll. 2.

1208. Inuento integrali particulari quocunque $y=V$ quod aequationi propoſitae ſatisfaciat, ſi ponamus deinceps $y=V+v$, orietur haec aequatio

$$0=\frac{nv}{a}-\frac{n(n-1)dv}{1.2a^2dx}+\frac{n(n-1)(n-2)ddv}{1.2.3a^3dx^2}-\text{etc.}$$

ex quo integrale completum erit

$$y=V+Ae^{2ax\,\text{fin.}\,\zeta^2}\,\text{fin.}(\alpha+ax\,\text{fin.}\,2\zeta)$$

vltima hac parte ſecundum omnes valores ipſius ζ multiplicata.

Coroll. 3.

1209. Si ſumamus $n=\infty$ et $a=n$, vt haec prodeat aequatio differentialis in infinitum excurrens

$$X=y-\frac{dy}{1.2dx}+\frac{ddy}{1.2.3dx^2}-\frac{d^3y}{1.2.3.4dx^3}+\frac{d^4y}{1.2.3.4.5dx^4}-\text{etc.}$$

erit y terminus ſummatorius progreſſionis, cuius terminus generalis indici x reſpondens eſt $T=\frac{dX}{dx}$.

Quam-

Quamdiu ergo angulus $\zeta = \frac{i\pi}{n}$ est infinite paruus ob $\Phi = 2i\pi x$ integralis pars quaelibet est:

$$4i\pi . e^{\frac{2ii\pi\pi}{n}x}\left(\sin\left(\tfrac{2i\pi}{n} - 2i\pi x\right)\int e^{\frac{-2ii\pi\pi}{n}x} X dx \cos(2i\pi x)\right.$$
$$\left. + \cos.\left(\tfrac{2i\pi}{n} - 2i\pi x\right)\int e^{\frac{-2ii\pi\pi}{n}x} X dx \sin.(2i\pi x)\right)$$

et omissis euanescentibus:

$$4i\pi(\cos.(2i\pi x)\int X dx \sin.(2i\pi x) - \sin.(2i\pi x)\int X dx \cos(2i\pi x))$$

si iam hic pro i successiue omnes numeri integri 1, 2, 3 etc. substituantur omnium formularum hoc modo resultantium summa dabit verum et completum valorem ipsius y.

Scholion.

1210. Pro aequatione autem proposita methodo ante indicata integrale particulare per seriem differentialium inuenire licet ponendo:

$$y = AX + \frac{BdX}{dx} + \frac{CddX}{dx^2} + \frac{Dd^3X}{dx^3} + \frac{Ed^4X}{dx^4} + \text{etc.}$$

facta enim substitutione reperitur:

$$A = \frac{a}{n};\ B = \frac{n-1}{2n};\ C = \frac{nn-1}{12an};\ D = \frac{nn-1}{24aan};$$
$$E = -\frac{(nn-1)(nn-19)}{720a^3n} \text{ etc.}$$

cuius quidem seriei difficile est legem progressionis in genere assignare. Verum pro casu $n = \infty$ et $a = n$, qui imprimis in Doctrina progressionum est notatu dignus, hi coefficientes ita se habent.

$$A = 1;\ B = \tfrac{1}{2};\ C = \tfrac{1}{12};\ D = 0;\ E = -\tfrac{1}{720} \text{ etc.}$$

vnde ea ipſa forma oritur, quam olim in genere pro termino ſummatorio dedi. Conceſſo autem hoc termino ſummatorio qui ſit $=V$ probe notari conuenit aequationem $y=V$ tantum eſſe integrale particulare aequationis propoſitae, completum vero facile exhiberi, ſi modo ad V addantur omnes huiusmodi formulae $A\,\text{ſin.}(\alpha+2i\pi x)$, pro i ſcribendo ſucceſſiue omnes numeros 1, 2, 3, 4 etc. vbi pro quolibet angulus α pro arbitrio aſſumi poteſt. Quod autem ſinguli hi valores aequationi

$$0=v-\frac{dv}{2dx}+\frac{ddv}{6dx^2}-\frac{d^3v}{24dx^4}+\frac{d^4v}{120dx^4}-\frac{d^5v}{720dx^5}+\text{etc.}$$

ſatisfaciant, ita facillime oſtenditur. Poſito breuitatis gratia $2i\pi=m$ vt ſit $v=\text{ſin.}(\alpha+mx)$ et facta ſubſtitutione fieri debet

$$0=\left.\begin{array}{l}\text{ſin.}(\alpha+mx)\left(1-\frac{mm}{6}+\frac{m^4}{120}-\text{etc.}\right)\\ \text{coſ.}(\alpha+mx)\left(-\frac{m}{2}+\frac{m^3}{24}-\frac{m^5}{720}+\text{etc.}\right)\end{array}\right\}=\begin{array}{l}\text{ſin.}(\alpha+mx)\frac{1}{m}\text{ſin.}m\\ \text{coſ.}(\alpha+mx)\frac{1}{m}(\text{coſ.}m-1).\end{array}$$

Cum autem ſit $m=2i\pi$ manifeſto eſt tam $\text{ſin.}m=0$ quam $\text{coſ.}m-1=0$.

Problema 161.

1211. Propoſita hac aequatione differentiali:

$$X=y+\frac{n(n-1)}{1\cdot 2}\cdot\frac{ddy}{a^2dx^2}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{d^4y}{a^4dx^4}+\text{etc.}$$

eius integrale completum inueſtigare.

Solutio.

Quantitas algebraica hinc formanda eſt

$$P=1+\frac{n(n-1)}{1\cdot 2}\cdot\frac{zz}{aa}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{z^4}{a^4}+\text{etc.}$$

quae

quae ad hanc formam manifesto reducitur:

$$P = \tfrac{1}{2}\left(1+\tfrac{z}{a}\right)^n + \tfrac{1}{2}\left(1-\tfrac{z}{a}\right)^n = \frac{(a+z)^n+(a-z)^n}{2a^n}$$

cuius factor quicunque trinomialis est

$$(a+z)^2 - 2(aa-zz)\operatorname{cof.} 2\zeta + (a-z)^2$$

sumendo

$$2\zeta = \frac{(2i+1)\pi}{n}, \text{ seu } \zeta = \frac{(2i+1)\pi}{2n}.$$

Haec autem forma abit in:

$$2aa(1-\operatorname{cof.} 2\zeta) + 2zz(1+\operatorname{cof.} 2\zeta) = 4aa \operatorname{fin.} \zeta^2 + 4zz \operatorname{cof.} \zeta^2$$

qui factor generalis repraesentetur hoc modo

$$aa \operatorname{tang.} \zeta^2 + zz$$

sicque comparatio cum forma generali

$$ff + 2fz \operatorname{cof.} \theta + zz$$

praebet

$$f = -a \operatorname{tang.} \zeta \text{ et } \theta = 90°,$$

vnde fit

$$\Phi = -ax \operatorname{tang.} \zeta \quad (1177.)$$

et valor pro z substituendus

$$-f(\operatorname{cof.} \theta + \sqrt{-1} . \operatorname{fin.} \theta) = a \operatorname{tang.} \zeta . \sqrt{-1}$$

quo pacto

$$\frac{dP}{dz} = \frac{n(a+z)^{n-1} - n(a-z)^{n-1}}{2a^n},$$

abire ponitur in $\mathfrak{P} + \mathfrak{Q}\sqrt{-1}$,

vnde

vnde fit:

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{n}{2a}((1+\text{tang.}\zeta.\sqrt{-1})^{n-1}-(1-\text{tang.}\zeta.\sqrt{-1})^{n-1})$$

$$=\frac{n}{2a\,\text{cof.}\zeta^{n-1}}(\text{cof.}(n-1)\zeta+\sqrt{-1}.\text{fin.}(n-1)\zeta-\text{cof.}(n-1)\zeta$$

$$+\text{fin.}(n-1)\zeta.\sqrt{-1})$$

ideoque $\mathfrak{P}=0$ et $\mathfrak{Q}=\frac{n\,\text{fin.}(n-1)\zeta}{a\,\text{cof.}\zeta^{n-1}}$. At ob $n\zeta=\frac{2i+1}{2}\pi$ hincque $\text{cof.}n\zeta=0$ et $\text{fin.}n\zeta=\pm 1$, prout i fuerit numerus par vel impar, erit $\text{fin.}(n-1)\zeta=\pm\text{cof.}\zeta$, ideoque $\mathfrak{Q}=\frac{\pm n}{a\,\text{cof.}\zeta^{n-2}}$. Quocirca ob $\text{cof.}\theta=0$, integralis pars ex hoc factore oriunda est:

$$\pm\frac{2a\,\text{cof.}\zeta^{n-2}}{n}(\text{cof.}\Phi\int X\,dx\,\text{fin.}\Phi-\text{fin.}\Phi\int X\,dx\,\text{cof.}\Phi)$$

seu ob $\Phi=-ax\,\text{tang.}\zeta$

$$\pm\frac{2a\,\text{cof.}\zeta^{n-2}}{n}(\text{fin.}(ax\,\text{tang.}\zeta)\int X\,dx\,\text{cof.}(ax\,\text{tang.}\zeta)$$

$$-\text{cof.}(ax\,\text{tang.}\zeta)\int X\,dx\,\text{fin.}(ax\,\text{tang.}\zeta))$$

vbi pro ζ successiue substituantur anguli

$$\frac{\pi}{2n};\ \frac{3\pi}{2n};\ \frac{5\pi}{2n};\ \frac{7\pi}{2n},$$

quamdiu sunt recto minores, pro quibus ibi alternatim + et − scribi oportet; haeque partes omnes in vnam summam collectae dabunt valorem completum ipsius y, dummodo pro vltima parte ex angulo $\zeta=\frac{\pi}{2}$ oriunda, quod euenit si n numerus impar, eius tantum semissis capiatur.

Coroll. 1.

Coroll. 1.

1212. Accommodemus haec ſtatim ad caſum $n = \infty$ et $a = nc$, vt propoſita ſit haec aequatio differentialis

$$X = y + \frac{ddy}{1 \cdot 2 c^2 dx^2} + \frac{d^4 y}{1 \cdot 2 \cdot 3 \cdot 4 c^4 dx^4} + \frac{d^6 y}{1 \ldots 6 c^6 dx^6} + \text{etc. in infinitum.}$$

Cum igitur hic valores ipſius ζ ſint infinite parui, erit

$$\text{coſ.}\,\zeta = 1 \quad \text{et} \quad \text{tang.}\,\zeta = \zeta = \frac{(4i \pm 1)\pi}{2n}$$

hinc $ax \,\text{tang.}\,\zeta = (4i \pm 1) c x \frac{\pi}{2}$

pro quo angulo ſcribamus ω. Ergo pars integralis quaecunque:

$$\pm 2c (\text{ſin.}\,\omega \textstyle\int X dx\, \text{coſ.}\,\omega - \text{coſ.}\,\omega \int X dx\, \text{ſin.}\,\omega)$$

vbi ſigna ambigua ſibi mutuo reſpondent.

Coroll. 2.

1213. Si tantum angulus $\frac{\pi}{2} c x$ ponatur $= \Phi$ integrale vniuerſum ita erit expreſſum:

$$\begin{aligned} \frac{y}{2c} = &+ \text{ſin}\,\Phi \textstyle\int X\, dx\, \text{coſ.}\,\Phi - \text{coſ.}\,\Phi \int X dx\, \text{ſin.}\,\Phi \\ &- \text{ſin.}\,3\Phi \textstyle\int X dx\, \text{coſ.}\,3\Phi + \text{coſ.}\,3\Phi \int X dx\, \text{ſin.}\,3\Phi \\ &+ \text{ſin.}\,5\Phi \textstyle\int X dx\, \text{coſ.}\,5\Phi - \text{coſ.}\,5\Phi \int X dx\, \text{ſin.}\,5\Phi \\ &- \text{ſin.}\,7\Phi \textstyle\int X dx\, \text{coſ.}\,7\Phi + \text{coſ.}\,7\Phi \int X dx\, \text{ſin.}\,7\Phi \\ &\text{etc.} \end{aligned}$$

quae formulae in infinitum ſunt continuandae.

Coroll. 3.

1214. Si ponamus $c = b\sqrt{-1}$, vt habeatur haec aequatio infinita:

$$X = y - \frac{ddy}{1\cdot 2\, b^2\, dx^2} + \frac{d^4 y}{1\cdots 4\, b^4\, dx^4} - \frac{d^6 y}{1\cdots 6\, b^6\, dx^6} + \text{etc.}$$

ac iam angulum $\frac{\pi}{2} bx$ vocemus ψ erit integrale completum

$$\begin{aligned} \frac{y}{b} = &+ e^{-\psi}\int e^{\psi} X dx - e^{\psi}\int e^{-\psi} X dx \\ &- e^{-3\psi}\int e^{3\psi} X dx + e^{3\psi}\int e^{-3\psi} X dx \\ &+ e^{-5\psi}\int e^{5\psi} X dx - e^{5\psi}\int e^{-5\psi} X dx \\ &\text{etc.} \end{aligned}$$

Scholion.

1215. Si pro aequatione Coroll. 1. methodo supra exposita quaeramus integrale particulare per differentialia ipsius X expressum huncque in finem ponamus;

$$y = AX - \frac{BddX}{c^2 dx^2} + \frac{Cd^4X}{c^4 dx^4} - \frac{Dd^6X}{c^6 dx^6} + \frac{Ed^8X}{c^8 dx^8} - \text{etc.}$$

reperiemus hos coefficientium valores:

$$A = 1;\ B = \frac{1}{1\cdot 2};\ C = \frac{5}{1\cdots 4};\ D = \frac{61}{1\cdots 6};\ E = \frac{1385}{1\cdots 8};$$
$$F = \frac{50521}{1\cdots 10}$$

Hicque valor si ponatur $= V$, vocato angulo $\frac{\pi}{2} cx = \Phi$ erit integrale completum:

$$y = V + A\sin.(\alpha + \Phi) + B\sin.(\beta + 3\Phi) + C\sin(\gamma + 5\Phi) + D\sin.(\delta + 7\Phi) \text{ etc.}$$

Proble-

Problema 162.

1216. Propofita aequatione differentiali:

$$X = y + \frac{n(n-1)}{1\cdot 2 a^2}\cdot\frac{dy}{dx} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4\, a^4}\cdot\frac{ddy}{dx^2} + \frac{n \ldots\ldots (n-5)}{1\ldots\ldots 6\, a^6}\cdot\frac{d^3 y}{dx^3} + \text{etc.}$$

eius integrale completum inueftigare.

Solutio.

Quantitas algebraica hinc formanda:

$$P = 1 + \frac{n(n-1)}{1\cdot 2\, a^2} z + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4\, a^4} zz + \text{etc.}$$

$$= \tfrac{1}{2}\left(1 + \tfrac{\sqrt{z}}{a}\right)^n + \tfrac{1}{2}\left(1 - \tfrac{\sqrt{z}}{a}\right)^n$$

cum ex praecedente nafcatur fi ibi loco zz fcribatur z, fumto angulo $\zeta = \frac{2\lambda+1}{2n}\pi$, factor quicunque erit $aa\,\text{tang.}\,\zeta^2 + z$, ita vt huius formae omnes factores fimplices fint reales. Hoc ergo factore cum formula $\alpha + z$ comparato, erit $\alpha = aa\,\text{tang.}\,\zeta^2$, et fumto $\mathfrak{A} = \frac{P}{\alpha + z}$ pofito $z = -\alpha$, erit integralis pars ex hoc factore oriunda $\frac{1}{\mathfrak{A}} e^{-\alpha x}\int e^{\alpha x} X\,dx$. Quia vero P euanefcit pofito $z = -\alpha$ erit quoque $\mathfrak{A} = \frac{dP}{dz}$; at eft differentiando:

$$\frac{dP}{dz} = \frac{n}{4a\sqrt{z}}\left(\left(1 + \frac{\sqrt{z}}{a}\right)^{n-1} - \left(1 - \frac{\sqrt{z}}{a}\right)^{n-1}\right)$$

Quia igitur poni oportet $\frac{\sqrt{z}}{a} = \text{tang.}\,\zeta\,\sqrt{-1}$ erit
$1 + \frac{\sqrt{z}}{a} = \frac{\text{cof.}\,\zeta + \sqrt{-1}.\,\text{fin.}\,\zeta}{\text{cof.}\,\zeta}$ et $1 - \frac{\sqrt{z}}{a} = \frac{\text{cof.}\,\zeta - \sqrt{-1}.\,\text{fin.}\,\zeta}{\text{cof.}\,\zeta}$

hincque

$$\mathfrak{A} = \frac{n}{4aa\,\text{tang.}\,\zeta.\sqrt{-1}}\cdot\frac{2\sqrt{-1}.\,\text{fin.}\,(n-1)\zeta}{\text{cof.}\,\zeta^{n-1}} = \frac{n\,\text{fin.}\,(n-1)\zeta}{2aa\,\text{fin.}\,\zeta\,\text{cof.}\,\zeta^{n-2}}$$

 Iam

Iam obferuetur effe $\sin. n\zeta = \sin.(2i+1)\frac{\pi}{2} = \pm 1$, (vbi fignum fuperius valet fi i numerus par, inferius fi impar) tum vero $\cos. n\zeta = 0$, vnde fit $\sin.(n-1)\zeta = \pm \cos.\zeta$, ex quo conficitur

$$\mathfrak{A} = \frac{\pm n}{2aa \sin.\zeta \cos.\zeta^{n-1}}$$

et pars integralis quaefita habebitur

$$\pm \frac{2aa \sin.\zeta \cos.\zeta^{n-1}}{n} e^{-aa\,\mathrm{tang}.\zeta^2.x} \int e^{aa\,\mathrm{tang}.\zeta^2.x} X\,dx.$$

Nunc igitur ipfi ζ fucceffiue tribuantur hi valores $\frac{\pi}{2n}$; $\frac{3\pi}{2n}$; $\frac{5\pi}{2n}$; $\frac{7\pi}{2n}$ etc. quoad angulum rectum non fuperent, atque omnes iftae partes in vnam fummam collectae, dabunt integrale completum feu valorem ipfius y.

Coroll. 1.

1217. Si ponamus $n = \infty$ et $a = nc$, aequatio propofita in infinitum excurrit, eritque

$$X = y + \frac{dy}{1.2c^2 dx} + \frac{ddy}{1.2.3.4c^4 dx^2} + \frac{d^3 y}{1\ldots 6c^6 dx^3} + \text{etc.}$$

et forma algebraica inde nata

$$P = 1 + \frac{z}{1.2c^2} + \frac{zz}{1.2.3.4c^4} + \frac{d^3}{1\ldots 6c^6} + \text{etc.} = \tfrac{1}{2} e^{\frac{\sqrt{z}}{c}} + \tfrac{1}{2} e^{\frac{-\sqrt{z}}{c}}$$

quae omnes factores fimplices habet reales, et ob ζ infinite paruum erit $\mathrm{tang}.\zeta = \zeta = \frac{2i+1}{2n}\pi$, indeque factorum forma generalis

$$z + \frac{(2i+1)^2}{4}\pi\pi cc \text{ feu } 1 + \frac{4z}{(2i+1)^2 \pi\pi cc}.$$

Coroll. 2.

Coroll. 2.

1218. Ponatur breuitatis gratia angulus

$$\frac{2i+1}{2}\pi = \theta \text{, erit}$$

$$aa\,\text{tang.}\,\zeta^2 = \theta\theta cc \text{, tum vero}$$

$$\text{cof.}\,\zeta = 1 \text{ et } \frac{aa\,\text{fin.}\,\zeta}{n} = \theta cc \text{,}$$

ex quo integralis pars quaecunque erit:

$$\pm 2\theta cce^{-\theta\theta ccx}\int e^{\theta\theta ccx}X\,dx$$

vbi pro θ fucceffiue omnes hos angulos fcribi oportet

$$\frac{\pi}{2};\ \frac{3\pi}{2};\ \frac{5\pi}{2};\ \frac{7\pi}{2};\ \frac{9\pi}{2} \text{ etc.}$$

Coroll. 3.

1219. Perinde hic eft fiue cc negatiue fiue pofitiue capiatur, hinc iftius aequationis differentialis infinitae

$$X = y + \frac{dy}{1.2\,bdx} + \frac{ddy}{1.2.3.4\,bbdx^2} + \frac{d^3y}{1\ldots 6\,b^3dx^3} + \text{etc.}$$

integrale erit

$$y = 2\theta be^{-\theta\theta bx}\int e^{\theta\theta bx}X\,dx$$

loco θ fcribendo fucceffiue omnes hos angulos, ambiguitate figni iam fublata

$$+\frac{\pi}{2};\ -\frac{3\pi}{2};\ +\frac{5\pi}{2};\ -\frac{7\pi}{2} \text{ etc.}$$

vnde fi $X = 0$ integrale particulare quoduis eft

$$y = Ae^{-\theta\theta bx}.$$

Problema 163.

1220. Propofita aequatione differentiali:

$$X = \frac{n\,dy}{a\,dx} + \frac{n(n-1)(n-2)ddy}{1.\;2.\;3\;a^3\,dx^3} + \frac{n(n-1)(n-2)(n-3)(n-4)d^5y}{1.\;2.\;3.\;4.\;5\;a^5\,dx^5} + \text{etc.}$$

eius integrale completum inueftigare.

Solutio.

Etfi haec aequatio in dx ducta fponte femel integratur, praeftat tamen hanc formam retinere, vnde fit

$$P = \frac{nz}{a} + \frac{n(n-1)(n-2)zz}{1.\;2.\;3\;a^3} + \frac{n(n-1)(n-2)(n-3)(n-4)z^3}{1.\;2.\;3.\;4.\;5\;a^5} + \text{etc.}$$

quae manifefto ita exhiberi poteft

$$P = \frac{\sqrt{z}}{2}\left(\left(1 + \frac{\sqrt{z}}{a}\right)^n - \left(1 - \frac{\sqrt{z}}{a}\right)^n\right)$$

cuius quidem ftatim vnus factor fe offert z; reliqui vero in hac forma continentur:

$$\left(1 + \frac{\sqrt{z}}{a}\right)^2 - 2\left(1 - \frac{z}{aa}\right)\text{cof.}\,2\zeta + \left(1 - \frac{\sqrt{z}}{a}\right)^2$$

fumto angulo

$$2\zeta = \frac{2i\pi}{n} \text{ feu } \zeta = \frac{i\pi}{n},$$

haec vero forma abit in

$$2(1 - \text{cof.}\,2\zeta) + \frac{2z}{aa}(1 + \text{cof.}\,2\zeta),$$

vnde patet in genere factorem fore $aa\,\text{tang.}\,\zeta^2 + z$, quae etiam primum illum z complectitur fumto $i = 0$. Hinc pofito $aa\,\text{tang.}\,\zeta^2 = \alpha$, integralis pars huic factori refpondens erit:

$$\frac{1}{\alpha}e^{-\alpha x}\int e^{\alpha x}X\,dx,$$

fi

si posito

$$z = -aa\,\text{tang.}\,\zeta^2 \quad \text{seu} \quad \sqrt{z} = a\,\text{tang.}\,\zeta\sqrt{-1},$$

capiatur

$$\mathfrak{A} = \frac{dP}{dx} = \frac{1}{4\sqrt{z}}\left(\left(1+\frac{\sqrt{z}}{a}\right)^n - \left(1-\frac{\sqrt{z}}{a}\right)^n\right) + \frac{n}{4a}\left(\left(1+\frac{\sqrt{z}}{a}\right)^{n-1} + \left(1-\frac{\sqrt{z}}{a}\right)^{n-1}\right)$$

at

$$\left(1+\frac{\sqrt{z}}{a}\right)^n = \frac{\text{cos.}\,n\zeta + \sqrt{-1}\,.\,\text{sin.}\,n\zeta}{\text{cos.}\,\zeta^n} \quad \text{et} \quad \left(1-\frac{\sqrt{z}}{a}\right)^n = \frac{\text{cos.}\,n\zeta - \sqrt{-1}\,.\,\text{sin.}\,n\zeta}{\text{cos.}\,\zeta^n},$$

quamobrem fiet

$$\mathfrak{A} = \frac{\text{sin.}\,n\zeta}{2a\,\text{tang.}\,\zeta\,\text{cos.}\,\zeta^n} + \frac{n\,\text{cos.}\,(n-1)\zeta}{2a\,\text{cos.}\,\zeta^{n-1}} = \frac{\pm n}{2a\,\text{cos.}\,\zeta^{n-2}}$$

ob $\text{sin.}\,n\zeta = 0$ et $\text{cos.}\,n\zeta = \pm 1$ prout numerus i fuerit vel par vel impar. Quocirca integralis pars quaecunque ita erit expressa:

$$\pm\frac{2a\,\text{cos.}\,\zeta^{n-2}}{n}\,e^{-\alpha x}\int e^{\alpha x}X\,dx$$

existente $\alpha = aa\,\text{tang.}\,\zeta^2$. Iam angulo ζ successiue tribuantur hi valores

$$\frac{0\pi}{n};\ \frac{1\pi}{n};\ \frac{2\pi}{n};\ \frac{3\pi}{n}\ \text{etc.}$$

quoad angulum rectum $\frac{\pi}{2}$ non excedant, haeque formulae omnes cum suis signis in vnam summam coniectae dabunt valorem completum pro y.

Coroll. 1.

1221. Prima igitur integralis pars nascitur ex angulo $\zeta = 0$, vnde ea erit $+\frac{2a}{n}\int X\,dx$, cuius autem

autem loco ob rationes ſupra allegatas circa factores ſimplices, eius tantum dimidium ſumi debet, vt haec prima pars ſit $=\frac{a}{n}\int X\,dx$, quod etiam inde patet, quod poſito $z=0$ fiat manifeſto $\frac{P}{z}=\frac{n}{a}$.

Coroll. 2.

1222. Idem tenendum eſſet de parte vltima ſiquidem ex valore $\zeta=\frac{\pi}{2}$ naſcatur, quod euenit ſi n ſit numerus par. Quia vero hoc caſu ſit $\cos.\zeta=0$, haec tota integralis pars per ſe euaneſcit.

Coroll. 3.

1223. Si eſſet $X=0$, quaelibet pars integralis foret $Ae^{-aa\,tang.\,\zeta^2\,x}$ denotante A quantitatem conſtantem arbitrariam foretque adeo haec aequatio

$$y=Ae^{-aa\,tang.\,\zeta^2\,x}$$

integrale particulare aequationis, dummodo capiatur angulus $\zeta=\frac{i\pi}{n}$.

Scholion.

1224. Hinc poſito $n=\infty$ et $a=n\sqrt{b}$ integrari poteſt haec aequatio differentialis in infinitum excurrens:

$$\frac{X}{\sqrt{b}}=\frac{dy}{1.b\,dx}+\frac{ddy}{1.2.3\,bb\,dx^2}+\frac{d^3y}{1\ldots5\,b^3dx^3}+\frac{d^4y}{1\ldots\ldots7\,b^4dx^4}+\text{etc.}$$

vel

vel etiam haec per vnam integrationem ex ista nata

$$\sqrt{b}.\int X\,dx = \frac{y}{1} + \frac{dy}{1.2.3.\,b\,dx} + \frac{ddy}{1\ldots 5\,b^2 dx^2} + \frac{d^3y}{1\ldots 7\,b^3 dx^3} + \text{etc.}$$

Cum enim sit angulus $\zeta = \frac{i\pi}{n}$ infinite paruus erit

$$\operatorname{cof.}\zeta = 1 \quad \text{et} \quad a\operatorname{tang.}\zeta = a\zeta = i\pi\sqrt{b},$$

ideoque

$$\alpha = aa\operatorname{tang.}\zeta^2 = ii\pi\pi b,$$

habebitur pars integralis quaecunque:

$$\pm 2\sqrt{b}.e^{-ii\pi\pi bx}\int e^{ii\pi\pi bx}X\,dx$$

vnde parte prima ex $i=0$ nata ad dimidium reducta ob rationes supra allegatas erit integrale completum

$$\frac{y}{\sqrt{b}} = \int X\,dx - 2e^{-\pi\pi bx}\int e^{\pi\pi bx}X\,dx + 2e^{-4\pi\pi bx}\int e^{4\pi\pi bx}X\,dx$$
$$- 2e^{-9\pi\pi bx}\int e^{5\pi\pi bx}X\,dx + 2e^{-16\pi\pi bx}\int e^{16\pi\pi bx}X\,dx \text{ etc.}$$

Exemplum.

1225. Sit $n=6$ et $a=1$, vt integranda proponatur haec aequatio:

$$X = \frac{6\,dy}{dx} + \frac{20\,ddy}{dx^2} + \frac{6\,d^3y}{dx^3}$$

seu

$$\int X\,dx = 6y + \frac{20\,dy}{dx} + \frac{6\,ddy}{dx^2}.$$

 Valo-

Valores ergo pro angulo ζ et inde pendentes ſunt

$$\zeta = 0\,,\ 30^\circ\,,\ 60^\circ$$

$$\mathrm{cof.}\,\zeta = 1\,,\ \tfrac{\sqrt{3}}{2}\,,\ \tfrac{1}{2}$$

$$a = 0\,,\ \tfrac{1}{3}\,,\ 3$$

ex quibus colligitur integrale quaeſitum:

$$y = \tfrac{1}{6}\int X\,dx - \tfrac{3}{16}e^{-\frac{1}{3}x}\int e^{\frac{1}{3}x}X\,dx + \tfrac{1}{48}e^{-3x}\int e^{3x}X\,dx$$

quod etiam aequationi ſatisfacere tentanti patebit.

CAPVT V.

CAPUT V.

DE INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM HUIUS FORMAE

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \frac{Ex^4d^4y}{dx^4} + \text{etc.}$$

Problema 164.

1226.

Proposita aequatione differentiali huius formae

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cxxddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \dots + \frac{Nx^nd^ny}{dx^n}$$

definire functionem ipsius x per quam ea multiplicata fiat integrabilis.

Solutio.

Attendenti mox patebit simplicem potestatem ipsius x hoc praestare. Sit igitur integrabilis haec aequatio:

$$Xx^\lambda dx = Ax^\lambda ydx + Bx^{\lambda+1}dy + \frac{Cx^{\lambda+2}ddy}{dx} + \dots + \frac{Nx^{\lambda+n}d^ny}{dx^{n-1}}$$

cuius

cuius integrale fit :

$$\int Xx^{\lambda}dx = A'x^{\lambda+1}y + \frac{B'x^{\lambda+2}dy}{dx} + \frac{C'x^{\lambda+3}ddy}{dx^2} + \ldots + \frac{M'x^{\lambda+n}d^{n-1}y}{dx^{n-1}}.$$

Cum igitur huius differentiale illi debeat esse aequale sequentes nanciscemur determinationes :

$A = (\lambda+1)A'$ hinc; $(\lambda+1)A' = A$

$B = (\lambda+2)B' + A'$; $(\lambda+1)(\lambda+2)B' = (\lambda+1)B - A$

$C = (\lambda+3)C' + B'$; $(\lambda+1)(\lambda+2)(\lambda+3)C' = (\lambda+1)(\lambda+2)C - (\lambda+1)B + A$

$D = (\lambda+4)D' + C'$; $(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)D' = (\lambda+1)(\lambda+2)(\lambda+3)D$
$- (\lambda+1)(\lambda+2)C + (\lambda+1)B - A$

.
.
.
.

$N = M'$

integralis enim termini sequentes, qui inuoluerent differenrentialis gradum $d^n y$ altioresque euanescere debent quia alioquin integratio non successisset. Cum igitur in integrali littera N' euanescat, peruenimus ad hanc aequationem :

$$0 = A - (\lambda+1)B + (\lambda+1)(\lambda+2)C - (\lambda+1)(\lambda+2)(\lambda+3)D \ldots \pm (\lambda+1)(\lambda+2) \ldots\ldots (\lambda+n)N$$

ex qua aequatione exponens λ potestatis quaesitae x^{λ} definiri debet. Formetur ergo talis expressio algebraica :

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \ldots \ldots + N(z-1)(z-2)(z-3)\ldots\ldots(z-n)$$

huius-

huiusque quaerantur omnes factores simplices vt sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z) \text{ etc.}$$

factorum horum numero existente $= n$. Iam ex quolibet factore $z + \alpha$ ad nihilum reducto valor $z = -\alpha$, dabit potestatem x^{α} per quam proposita aequatio multiplicata integrabilis euadit, ita vt eius integrale sit futurum

$$x^{-\alpha-1}\int x^{\alpha} X dx = A'y + \frac{B'x dy}{dx} + \frac{C'x^2 ddy}{dx^2} + \frac{D'x^3 d^3 y}{dx^3} + \dots + \frac{Nx^{n-1} d^{n-1} y}{dx^{n-1}}$$

vbi differentialium gradus vnitate est inferior. Ita autem haec aequatio integrata, per propositam determinatur, vt sit

$$A = (\alpha + 1)A'$$
$$B = (\alpha + 2)B' + A'$$
$$C = (\alpha + 3)C' + B'$$
$$D = (\alpha + 4)D' + C'$$
etc.

donec perueniatur ad vltimum coefficientem N qui vtrobique est idem.

Coroll. 1.

1227. Quia aequatio integrata similis est ipsi propositae, ea per certam potestatem ipsius x multiplicata denuo fiet integrabilis. Ad hanc enim pote-

poteſtatem inueniendam conſiderare oportet hanc formam algebraicam:

$$Q = A' + B'(z-1) + C'(z-1)(z-2) + D'(z-1)(z-2)(z-3) \ldots$$
$$\ldots + N(z-1)(z-2) \ldots (z-n+1)$$

cuius ſi fuerit factor ſimplex quicunque $z+\mu$ erit $x^{+\mu}$ illa poteſtas ipſius x, aequationem integrabilem reddens.

Coroll. 2.

1228. Quodſi ipſa aequatio propoſita per poteſtatem $x^{+\alpha}$ multiplicata reddita fuerit integrabilis, hic probe notari conuenit quantitatem Q ex integrata formatam ita pendere a priori P ex ipſa propoſita formata vt ſit $Q = \frac{P}{\alpha + z}$, quandoquidem per hypotheſin $\alpha + z$ eſt factor ipſius P.

Scholion 1.

1229. Ad hanc inſignem proprietatem demonſtrandam quod ſcilicet ſit $P = (\alpha + z)Q$, tantum opus eſt vt quantitas Q per $\alpha + z$ multiplicetur; verum quo concluſio clarius in oculos occurrat pro ſingulis terminis ipſius Q multiplicator bipartito eſt repraeſentandus, ac pro primo quidem termino loco $\alpha + z$ ſcribatur $(\alpha+1)+(z-1)$ pro ſecundo $(\alpha+2)+(z-2)$ pro tertio $(\alpha+3)+(z-3)$ pro quarto $(\alpha+4)+(z-4)$ etc. ita vt cuiusque termini productum binis parti-

bus

bus exhibeatur, quam operationem hic totam apponam:

$$Q = A' + B'(z-1) + C'(z-1)(z-2) + D'(z-1)(z-2)(z-3) \text{ etc.}$$

Multipl.				
	$\alpha+1$	$\alpha+2$	$\alpha+3$	$\alpha+4$
	$z-1$	$z-2$	$z-3$	$z-4$

$$\text{Prod. } (\alpha+1)A' + A'(z-1) \quad + B'(z-1)(z-2) \quad + C'(z-1)(z-2)(z-3)$$
$$+(\alpha+2)B'(z-1) + (\alpha+3)C'(z-1)(z-2) + (\alpha+4)D'(z-1)(z-2)(z-3) \text{ etc.}$$

In solutione autem vidimus esse:

$$(\alpha+1)A' = A\,;\quad (\alpha+2)B' + A' = B\,;\quad (\alpha+3)C' + B' = C \text{ etc.}$$

quocirca hoc productum hac forma exprimetur

$$A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) \text{ etc.}$$

cui valor ipsius P est aequalis, sicque demonstrata est insignis illa proprietas memorata quod sit $Q = \frac{P}{\alpha+z}$.

Coroll. 3.

1230. Quodsi ergo valor ipsius P in factores simplices resolutus ita repraesentetur:

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z) \text{ etc.}$$

et ex factore $\alpha+z$ aequatio proposita per x^{α} multiplicata integretur, tum vero ex integrata simili modo valor Q formetur, erit

$$Q = N(\beta+z)(\gamma+z)(\delta+z) \text{ etc.}$$

Coroll. 4.

Coroll. 4.

1231. Aequatio ergo integrata, postquam ad formam propositae fuerit perducta, vt posito

$$x^{-\alpha-1}\int x^\alpha X\,dx = X'$$

habeatur

$$X' = A'y + \frac{B'x\,dy}{dx} + \frac{C'x^2\,ddy}{dx^2} + \frac{D'x^3\,d^3y}{dx^3} + \text{etc.}$$

haec denuo integrabilis reddetur, si multiplicetur per quampiam harum potestatum x^β, x^γ, x^δ etc. quae etiam ipsam propositam integrabilem reddidissent.

Scholion 2.

1232. Antequam continuationem harum integrationum vlterius prosequar, conueniet eum casum formae propositae generalis seorsim euolui, quo prius aequationis membrum X in nihilum abit. Hoc enim casu hoc commodi vsu venit, vt statim sine integrationibus repetitis integrale completum exhiberi queat, idque simili modo quo supra in Cap. II. sum vsus. Huic quidem casui quia iam multo facilius tractari potest, proprium caput assignare nolui, ne praecepta nimis multiplicari videantur.

Problema 165.

1233. Proposita hac aequatione differentiali cuiuscunque ordinis:

$$0 = Ay + \frac{Bx\,dy}{dx} + \frac{Cx^2\,ddy}{dx^2} + \frac{Dx^3\,d^3y}{dx^3} + \text{etc.}$$

vbi

vbi variabilis y cum suis differentialibus nusquam plus vna dimensione, altera vero x adeo nullam obtineat, eius integrale completum inuenire.

Solutio.

Particulariter huic aequationi satisfieri perspicicuum est, si y certae ipsius x potestati aequetur, ponamus ergo esse $y = x^{\mu}$, et facta substitutione, cum vbique per x^{μ} diuiserimus, perueniemus ad hanc aequationem

$$0 = A + \mu B + \mu(\mu - 1)C + \mu(\mu - 1)(\mu - 2)D + \text{etc.}$$

vnde exponentem μ determinari oportet. Vel si secundum praeceptum praec. Probl. ex aequatione proposita hanc formemus formulam algebraicam:

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) \text{ etc.}$$

eamque in factores simplices resoluamus, vt sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z) \text{ etc.}$$

euidens est posito $\mu = z - 1$, primae aequationi satisfieri sumendo

$$\mu = -\alpha - 1, \text{ vel } \mu = -\beta - 1, \text{ vel } \mu = -\gamma - 1 \text{ etc.}$$

ita vt quisque factor suppeditet integrale particulare. Cum igitur factorum numerus aequetur gradui differentialium summo, hinc colligetur integrale completum aequationis propositae

$$y = \mathfrak{A} x^{-\alpha-1} + \mathfrak{B} x^{-\beta-1} + \mathfrak{C} x^{-\gamma-1} + \mathfrak{D} x^{-\delta-1} \text{ etc.}$$

vbi tantum obferuari conuenit, fi factorum illorum fimplicium duo pluresue fuerint inter fe aequales, integralis formam fimili modo immutari debere quo fupra Cap. II. fum vfus. Scilicet cum aequationes ibi tractatae ad praefentem formam reuocentur, fi ibi loco x fcribatur lx, inde has regulas haurimus.

1) Si forma P factorem habeat $(\alpha+z)^2$, inde nafcitur pars integralis

$$=x^{-\alpha-1}(\mathfrak{A}+\mathfrak{B}lx).$$

2) Si forma P factorem habeat $(\alpha+z)^3$ pars integralis inde orta eft

$$x^{-\alpha-1}(\mathfrak{A}+\mathfrak{B}lx+\mathfrak{C}(lx)^2).$$

3) Si forma P factorem habeat $(\alpha+z)^4$ pars integralis inde orta erit

$$=x^{-\alpha-1}(\mathfrak{A}+\mathfrak{B}lx+\mathfrak{C}(lx)^2+\mathfrak{D}(lx)^3).$$

Si factores occurrant imaginarii partes inde oriundae per folitam imaginariorum reductionem facile ad formam realem reuocabuntur, vti in corollariis docebo.

Coroll. 1.

1234. Si forma P duos habeat factores fimplices imaginarios in formula $ff+2fz\,\text{cof.}\,\theta+zz$ contentos, hac cum producto $(\alpha+z)(\beta+z)$ comparata erit

$\alpha=f(\text{cof.}\,\theta+\sqrt{-1}.\,\text{fin.}\,\theta)$ et $\beta=f(\text{cof.}\,\theta-\sqrt{-1}.\,\text{fin.}\,\theta)$

vnde

vnde fit

$$x^{-\alpha} = x^{-f\cos.\theta} x^{-f\sqrt{-1}.\sin.\theta} = x^{-f\cos.\theta} e^{-\sqrt{-1}.f\sin.\theta lx}.$$

Est vero

$$e^{-u\sqrt{-1}} = \cos.u - \sqrt{-1}.\sin.u\,,$$

ideoque habetur

$$x^{-\alpha-1} = x^{-f\cos.\theta}\left(\frac{\cos.(f\sin.\theta\, lx) - \sqrt{-1}.\sin.(f\sin.\theta\, lx)}{x}\right).$$

Quare cum $x^{-\beta-1}$ simili modo exprimatur mutato signo ipsius $\sqrt{-1}$, ex factore duplici $ff+2fz\cos.\theta+zz$ haec nascitur pars integralis:

$$x^{-f\cos.\theta-1}(\mathfrak{A}\cos.(f\sin.\theta\, lx)+\mathfrak{B}\sin.(f\sin.\theta\, lx))$$

quae etiam ita potest repraesentari

$$\mathfrak{A}x^{-f\cos.\theta-1}\cos.(\mathfrak{a}+f\sin.\theta\, lx)\,,$$

denotante $\mathfrak{a}$ angulum constantem arbitrarium.

Coroll. 2.

1235. Simili modo si forma P duos huiusmodi factores aequales inuoluat, vt sit

$$(\alpha+z)^2(\beta+z)^2 = (ff+2fz\cos.\theta+zz)^2\,,$$

litterae α et β, eosdem quos ante sortientur valores imaginarios ex quorum reductione colligitur haec pars integralis inde oriunda:

$$x^{-f\cos.\theta-1}(\mathfrak{A}\cos.(\mathfrak{a}+f\sin.\theta lx)+\mathfrak{B}lx.\cos.(\mathfrak{b}+f\sin.\theta lx))$$

quatuor constantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{a}$ et $\mathfrak{b}$ continens.

Coroll. 3.

1236. Hinc ergo euidens eſt, quomodo ex factoribus formae P, ſiue ſint ſimplices ſiue duplices; ſiue inaequales ſiue aequales ſingulas integralis partes aſſignari indeque totum integrale completum formari conueniat.

Scholion.

1237. Totum ergo negotium huc redit, vt quantitas algebraica ex aequatione differentiali formata:

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$$

in ſuos factores reales vel ſimplices vel duplices reſoluetur in quo plerumque maxima difficultas verſatur, quoniam huiusmodi formae minus tractari ſunt ſolitae. Cum vero haec reſolutio iſti aequationi cum generali, quam hoc capite euoluere ſuſcepi eſt communis, quicquid hic praeſtare licuerit potius in aequatione generali oſtendi conueniet ad quam reſoluendam propterea reuertor. Id tantum hic obſeruaſſe neceſſe duco, quod ſi pro aequatione generali

$$X = Ay + \frac{Bx\,dy}{dx} + \frac{Cxx\,ddy}{dx^2} + \frac{Dx^3\,d^3y}{dx^3} + \text{etc.}$$

vndecunque innotuerit integrale particulare puta $y = V$ exiſtente V certa functione ipſius x, tum poſito $y = V + v$ perueniri ad hanc aequationem:

$$0 = Av + \frac{Bx\,dv}{dx} + \frac{Cxx\,ddv}{dx^2} + \frac{Dx^3\,d^3v}{dx^3} + \text{etc.}$$

cuius

cuius integrale completum per praecepta huius problematis inuentum si loco v scribatur, habebitur integrale completum illius aequationis, quo pacto certe insigne calculi compendium obtinetur.

Problema 166.

1238. Proposita aequatione differentiali gradus cuiuscunque n huius formae:

$$X = Ay + \frac{Bx\,dy}{dx} + \frac{Cx^2\,ddy}{dx^2} + \ldots\ldots + \frac{Nx^n d^n y}{dx^n}$$

eius integrale per integrationem n vicibus repetitam inuenire.

Solutio.

Ex hac aequatione formetur haec quantitas algebraica

$$P = A + B(z-1) + C(z-1)(z-2) + \ldots + N(z-1)(z-2)\ldots(z-n)$$

cuius quaerantur omnes factores simplices nullo habito respectu siue sint reales siue imaginarii, vt ea hoc modo exprimatur:

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\ldots\ldots(\mu+z)(\nu+z)$$

factorum numero existente $= n$. Quo facto, initio huius capitis vidimus quemlibet factorem puta $\alpha+z$ praebere potestatem x^α, per quam nostra aequatio fiat integrabilis, atque adeo ostendimus, integrale inde ortum, si compendii gratia ponamus

$$x^{-\alpha-1}\int x^\alpha X\,dx = X'$$

 fore

fore

$$X' = A'y + \frac{B'xdy}{dx} + \frac{C'x^2ddy}{dx^2} + \ldots\ldots + \frac{Nx^{n-1}d^{n-1}y}{dx^{n-1}}$$

ita vt fit $A' = \frac{A}{\alpha+1}$, ceterique coefficientes ita se habeant vti ibidem docuimus, hic autem sufficiet ad primum potissimum respexisse. Absoluta iam prima integratione si eadem lege ex aequatione semel integrata formemus quantitatem:

$$P' = A' + B'(z-1) + C'(z-1)(z-2) \ldots + N(z-1)(z-2) \ldots (z-n+1)$$

cuius resolutio in factores iam ex prima forma P constat postquam §. 1229. demonstraui esse

$$P' = N(\beta+z)(\gamma+z)(\delta+z) \ldots (\mu+z)(\nu+z)$$

ita vt fit $P' = \frac{P}{\alpha+z}$. Hinc ergo simili modo factor $\beta+z$ suppeditabit multiplicatorem x^β quo haec aequatio integrabilis redditur, ac posito

$$x^{-\beta-1}\int x^\beta X' dx = X''$$

vt fit

$$X'' = x^{-\beta-1}\int x^{\beta-\alpha-1} dx \int x^\alpha X dx$$

integrale erit

$$X'' = A''y + \frac{B''xdy}{dx} + \frac{C''x^2ddy}{dx^2} + \ldots + \frac{Nx^{n-2}d^{n-2}y}{dx^{n-2}}$$

existente hic $A'' = \frac{A'}{\beta+1} = \frac{A}{(\alpha+1)(\beta+1)}$. Quodsi hoc modo tot integrationes successiue absoluantur, quot vnitates continentur in indice n, sicque omnes factores simplices formae P in vsum vocentur, tandem ad

ad hanc peruenietur aequationem: $X^{(n)} = A^{(n)} y$, quae est ipsa integralis desiderata. Cum autem hic futurum sit

$$A^{(n)} = \frac{A}{(\alpha+1)(\beta+1)(\gamma+1)\ldots\ldots(\nu+1)}$$

euidens est denominatorem nasci ex forma $\frac{P}{N}$ si loco z scribatur vnitas; tum autem sumto $z = 1$, prima forma manifesto dat $P = A$, ita vt denominator iste fiat $= \frac{A}{N}$ ideoque $A^{(n)} = N$, quod etiam inde patet, quod omnium aequationum vltimi termini habeant eundem coefficientem N, quo ergo in postrema integrali ipse primus terminus y debet esse affectus. Deinde vero est

$$X^{(n)} = x^{-\nu-1} \int x^{\nu-\mu-1} dx \int x^{\mu-\lambda-1} dx \ldots\ldots \int x^{\beta-\alpha-1} dx \int x^{\alpha} X dx$$

vbi cum numeri α, β, γ etc. vtcunque inter se permutari possunt, integrale quaesitum etiam hoc modo repraesentari potest:

$$Ny = x^{-\alpha-1} \int x^{\alpha-\beta-1} dx \int x^{\beta-\gamma-1} dx \ldots\ldots \int x^{\mu-\nu-1} dx \int x^{\nu} X dx.$$

Coroll. 1.

1239. Totum ergo negotium huc redit, vt forma algebraica

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) \text{ etc.}$$

in suos factores simplices resoluatur, quibus inuentis vt sit

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\ldots\ldots(\nu+z)$$

hinc

hinc integrale quaesitum facile exhibetur, et quidem pro factorum varia permutatione pluribus modis, qui autem omnes eundem valorem exprimunt, vti ex sequentibus clarius patebit.

Coroll. 2.

1240. Cum haec forma integralis inuenta tot inuoluat integrationes, quoti gradus fuerit aequatio differentialis proposita, totidem quoque constantes arbitrariae ingerentur, quemadmodum indoles integrationis completae postulat.

Scholion.

1241. Quoniam integrale inuentum pluribus integrationibus est inuolutum, ad vsum faciliorem conueniet hanc formam in partes resolui, quae singulae vnicum tantum signum integrale contineant. Hanc autem resolutionem simili modo instituere licet, quo supra sumus vsi, atque hic quidem totum negotium ad huiusmodi formulam reuocatur

$$\int x^{m-n-1}dx\int x^n X dx,$$

quae manifesto ita reducitur, vt sit

$$\frac{1}{m-n}x^{m-n}\int x^n X dx - \frac{1}{m-n}\int x^m X dx$$

vbi tamen obseruandum est si fuerit $m=n$ peculiari reductione opus esse, hocque casu fore:

$$\int\frac{dx}{x}\int x^n X dx = lx.\int x^n X dx - \int x^n X dx\, lx.$$

Hac.

Hac ergo regula vtemur in resolutione sequentium problematum, quibus successiue omnes gradus differentialium percurramus, praetermisso quidem gradu primo, cum aequationis

$$X = Ay + \frac{Nx\,dy}{dx}, \text{ ob}$$

$$P = A + N(z-1) = N(\alpha + z)$$

integrale sit

$$Ny = x^{-\alpha-1}\int x^{\alpha} X\,dx$$

quod nulla reductione indiget.

Problema 167.

1242. Proposita hac aequatione differentiali secundi gradus:

$$X = Ay + \frac{Bx\,dy}{dx} + \frac{Nx^2\,ddy}{dx^2}$$

eius integrale per formulas integrales simplices euoluere.

Solutio.

Cum sit

$$P = A + B(z-1) + N(z-1)(z-2)$$

statuatur

$$P = N(\alpha + z)(\beta + z)$$

eritque integrale per methodum praecedentem inventum $Ny = X''$ existente

$$x^{\beta+1} X'' = \int x^{\beta-\alpha-1}\,dx \int x^{\alpha} X\,dx,$$

 quae

quae forma euoluitur in hanc

$$\frac{1}{\beta-\alpha}x^{\beta-\alpha}\int x^{\alpha}X\,dx-\frac{1}{\beta-\alpha}\int x^{\beta}X\,dx$$

ficque erit

$$Ny=\frac{x^{-\alpha-1}}{\beta-\alpha}\int x^{\alpha}X\,dx+\frac{x^{-\beta-1}}{\alpha-\beta}\int x^{\beta}X\,dx.$$

Hinc autem cafum excipi oportet, quo $\beta=\alpha$, tum enim fit

$$x^{\alpha+1}X''=\int\frac{dx}{x}\int x^{\alpha}X\,dx=lx\int x^{\alpha}X\,dx-\int x^{\alpha}X\,dx\,lx$$

pro hoc igitur cafu habebimus:

$$Ny=x^{-\alpha-1}\int\frac{dx}{x}\int x^{\alpha}X\,dx \text{ feu}$$

$$Ny=x^{-\alpha-1}(lx\int x^{\alpha}X\,dx-\int x^{\alpha}X\,dx\,lx)$$

vbi quidem prior forma praeferenda videtur.

Coroll. I.

1243. Si ambo factores fimplices fint imaginarii, ponatur

$$(\alpha+z)(\beta+z)=ff+2fz\,\text{cof.}\,\theta+zz,$$

eritque

$$\alpha=f(\text{cof.}\,\theta+\sqrt{-1}.\text{fin.}\,\theta) \text{ et}$$

$$\beta=f(\text{cof.}\,\theta-\sqrt{-1}.\text{fin.}\,\theta)$$

indeque

$$\beta-\alpha=-2f\sqrt{-1}.\text{fin.}\,\theta.$$

Tum

Tum vero

$x^{\alpha} = x^{f\,cof.\,\theta}(\text{cof.}(f\,\text{fin.}\,\theta.\,lx) + \sqrt{-1}.\,\text{fin.}(f\,\text{fin.}\,\theta.\,lx))$ et

$x^{-\alpha} = x^{-f\,cof.\,\theta}(\text{cof.}(f\,\text{fin.}\,\theta.\,lx) - \sqrt{-1}.\,\text{fin.}(f\,\text{fin.}\,\theta.\,lx)$

quae formulae mutato figno ipfius $\sqrt{-1}$ ad x^{β} et $x^{-\beta}$ transferuntur.

Coroll. 2.

1244. Ponatur breuitatis gratia angulus

$$f\,\text{fin.}\,\theta.\,lx = \Phi$$

et facta fubftitutione habebimus:

$$Nxy = \frac{x^{-f\,cof.\,\theta}(\text{cf.}\Phi - \sqrt{-1}.\,\text{fi.}\Phi)\int x^{f\,cof.\,\theta} X dx(\text{cf.}\Phi + \sqrt{-1}.\,\text{fi.}\Phi)}{-2f\sqrt{-1}.\,\text{fin.}\,\theta}$$
$$+ \frac{x^{-f\,cof.\,\theta}(\text{cf.}\Phi + \sqrt{-1}.\,\text{fi.}\Phi)\int x^{f\,cof.\,\theta} X dx(\text{cf.}\Phi - \sqrt{-1}.\,\text{fi.}\Phi)}{2f\sqrt{-1}.\,\text{fin.}\,\theta}$$

vbi partes imaginariae fe fponte deftruunt fietque

$$Nxy = \frac{x^{-f\,cof.\,\theta}}{f\,\text{fin.}\,\theta}(\text{fin.}\,\Phi\int x^{f\,cof.\,\theta} X dx\,\text{cof.}\,\Phi - \text{cof.}\,\Phi\int x^{f\,cof.\,\theta} X dx\,\text{fin.}\,\Phi).$$

Coroll. 3.

1245. Forma ergo haec realis modo inuenta aequiualet illi imaginaria implicanti

$$Nxy = \frac{x^{-\alpha}}{\beta - \alpha}\int x^{\alpha} X dx + \frac{x^{-\beta}}{\alpha - \beta}\int x^{\beta} X dx$$

fi fuerit,

$$(\alpha + z)(\beta + z) = ff + 2fz\,\text{cof.}\,\theta + zz,$$

 pona-

Eſt vero

$$\frac{1}{(\beta-\alpha)(\gamma-\alpha)}+\frac{1}{(\alpha-\beta)(\gamma-\beta)}=\frac{-1}{(\alpha-\gamma)(\beta-\gamma)}$$

quod quemadmodum cum per ſe liquet, tum vero ex Theoremat. §. 1169. demonſtrato perſpicitur. Quocirca integrale quaeſitum ita obtinetur expreſſum:

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}Xdx}{(\beta-\alpha)(\gamma-\alpha)}+\frac{x^{-\beta}\int x^{\beta}Xdx}{(\alpha-\beta)(\gamma-\beta)}+\frac{x^{-\gamma}\int x^{\gamma}Xdx}{(\alpha-\gamma)(\beta-\gamma)}.$$

Coroll. 1.

1247. Si forma P duos habeat factores aequales, vt ſit $\beta=\alpha$, quia tum eſt

$$X''=x^{-\alpha-1}\int\frac{dx}{x}\int x^{\alpha}Xdx \text{ erit}$$

$$x^{\gamma}X''dx=\frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx-\frac{1}{\gamma-\alpha}\int x^{\gamma-\alpha-1}dx\int x^{\alpha}Xdx$$

at $\int x^{\gamma-\alpha-1}dx\int x^{\alpha}Xdx=\frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int x^{\alpha}Xdx-\frac{1}{\gamma-\alpha}\int x^{\gamma}Xdx$

vnde colligitur:

$$Nxy=\frac{x^{-\alpha}}{\gamma-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx-\frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int x^{\alpha}Xdx+\frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int x^{\gamma}Xdx.$$

Coroll. 2.

1248. Similis forma oritur, ſi ſumatur $\gamma=\beta$, tum enim fit:

$$\int x^{\gamma}X''dx=\frac{x^{\beta-\alpha}}{(\beta-\alpha)^2}\int x^{\alpha}Xdx-\frac{\int x^{\beta}Xdx}{(\beta-\alpha)^2}+\frac{1}{\alpha-\beta}\int\frac{dx}{x}\int x^{\beta}Xdx$$

 ideo-

ideoque

$$Nxy = \frac{x^{-\beta}}{\alpha-\beta}\int\frac{dx}{x}\int x^{\beta}X\,dx - \frac{x^{-\beta}\int x^{\beta}X\,dx}{(\beta-\alpha)^2} + \frac{x^{-\alpha}\int x^{\alpha}X\,dx}{(\beta-\alpha)^2}$$

Coroll. 3.

1249. Quodsi autem omnes tres factores inter se fuerint aequales $\alpha=\beta=\gamma$ erit

$$\int x^{\gamma}X''dx = \int\frac{dx}{x}\int\frac{dx}{x}\int x^{\alpha}X\,dx$$

ideoque hoc casu integrale ita succincte exprimitur:

$$Nxy = x^{-\alpha}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\alpha}X\,dx.$$

Coroll. 4.

1250. Si duo factores sint imaginarii scilicet

$$(\alpha+z)(\beta+z) = ff + 2fz\cos.\theta + zz \text{ ob}$$

$$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta) \text{ et } \beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta)$$

postremum quidem nostri integralis membrum manet reale ob

$$(\alpha-\gamma)(\beta-\gamma) = \gamma\gamma - 2\gamma f\cos.\theta + ff,$$

at bina priora fient posito $\Phi = f\sin.\theta\, lx$

$$\frac{x^{-f\cos.\theta}(\cos.\Phi-\sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta}X\,dx(\cos.\Phi+\sqrt{-1}.\sin.\Phi)}{-2f\sqrt{-1}.\sin.\theta(\gamma-f(\cos.\theta+\sqrt{-1}.\sin.\theta))}$$

$$+\frac{x^{-f\cos.\theta}(\cos.\Phi+\sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta}X\,dx(\cos.\Phi-\sqrt{-1}.\sin.\Phi)}{2f\sqrt{-1}.\sin.\theta(\gamma-f(\cos.\theta-\sqrt{-1}.\sin.\theta))}$$

quae

quae reducuntur ad hanc formam realem

$$\frac{x^{-f\cos.\theta}(\gamma\,\mathrm{fi}.\Phi - f\,\mathrm{fi}.(\theta+\Phi))\int x^{f\cos.\theta}X\,dx\,\mathrm{cf}.\Phi - x^{-f\cos.\theta}(\gamma\,\mathrm{cf}.\Phi - f\,\mathrm{cf}.(\theta+\Phi))\int x^{f\cos.\theta}X\,dx\,\mathrm{fi}.\Phi}{f\,\mathrm{fin}.\theta\,(\gamma\gamma - 2\gamma f\,\mathrm{cof}.\theta + ff)}.$$

Scholion.

1251 Quod ad factores imaginarios attinet, integralium inde natorum reductio facilius in genere instituetur, vnde in his differentialium gradibus determinatis, ei non amplius immorabor. Factores autem aequales hic data opera pro fingulis gradibus accuratius perfequi eft vifum, quia fupra nimis cito ad euolutionem generalem properanti in infignem errorem illabi contigit, quem ftatim feliciter euitaffem; fi eadem methodo ibi effem vfus. Huiusmodi autem vitium circa factores imaginarios hic non eft pertimefcendum, cum in hoc negotio nihil fub fpecie infinite parui negligendum occurrat. Ex hoc autem fonte errores illi, quos fupra commifi funt nati, quod vitium fubtile quo clarius ob oculos ponatur, vna cum necefaria emendatione hic euoluam. Quaeftio fcilicet pro cafu praefenti huc redit, vt valor harum duarum formularum

$$\frac{x^{-\alpha}\int x^{\alpha}X\,dx}{(\beta-\alpha)(\gamma-\alpha)} + \frac{x^{-\beta}\int x^{\beta}X\,dx}{(\alpha-\beta)(\gamma-\beta)}$$

definiatur cafu quo $\beta=\alpha$, et ambo membra in infinitum excrefcent; hunc in finem pono $\beta=\alpha+\omega$ exiftente ω particula euanefcente et cum fit

$$x^{\beta}=x^{\alpha}x^{\omega}=x^{\alpha}e^{\omega lx}=x^{\alpha}(1+\omega lx),$$

hinc-

hincque

$$x^{-\beta} = x^{-\alpha}(1 - \omega\, lx),$$

habebimus:

$$\frac{x^{-\alpha}\int x^\alpha X\,dx}{\omega(\gamma-\alpha)} - \frac{x^{-\alpha}(1-\omega lx)\int x^\alpha X\,dx(1+\omega lx)}{\omega(\gamma-\beta)}.$$

Quia nunc est

$$\frac{1}{\gamma-\alpha} = \frac{1}{\gamma-\beta+\omega} = \frac{1}{\gamma-\beta} - \frac{\omega}{(\gamma-\beta)^2},$$

prius membrum induit hanc formam

$$\frac{x^{-\alpha}\int x^\alpha X\,dx}{\omega(\gamma-\beta)} - \frac{x^{-\alpha}\int x^\alpha X\,dx}{(\gamma-\beta)^2}$$

posterius vero euolutum istam

$$\frac{-x^{-\alpha}\int x^\alpha X\,dx}{\omega(\gamma-\beta)} + \frac{x^{-\alpha}lx\int x^\alpha X\,dx - x^{-\alpha}\int x^\alpha X\,dx\,lx}{\gamma-\beta}$$

sicque valor quaesitus casu $\beta = \alpha$ concluditur

$$\frac{x^{-\alpha}(lx\int x^\alpha X\,dx - \int x^\alpha X\,dx\,lx)}{\gamma-\alpha} - \frac{x^{-\alpha}\int x^\alpha X\,dx}{(\gamma-\alpha)^2} \quad \text{seu}$$

$$\frac{x^{-\alpha}}{\gamma-\alpha}\int\frac{dx}{x}\int x^\alpha X\,dx - \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int x^\alpha X\,dx$$

cuius formulae posterius membrum, quod in vitiosa illa methodo erat omissum, inde resultat, quod ad discrimen hic inter expressiones $\gamma-\alpha$ et $\gamma-\beta$ respeximus, quam necessariam cautionem supra neglEximus.

Problema

Problema 169.

1252. Propofita hac aequatione differentiali quarti gradus:

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cxxddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \frac{Nx^4d^4y}{dx^4},$$

eius integrale per formulas integrales fimplices euolvere.

Solutio.

Formata hinc expreffione algebraica:

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) \\ + N(z-1)(z-2)(z-3)(z-4)$$

ftatuatur

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)$$

et per praecepta generalia eft

$$Ny = X^{IV} \text{ exiftente } X^{IV} = x^{-\delta-1}\int X''' x^\delta dx;$$

fiquidem X''' ex tribus prioribus factoribus determinetur, quemadmodum in problemate praecedente eft factum. Valorem fcilicet ibi pro Nxy inuentum hic per $x^{\delta-1}dx$ multiplicari oportet, vnde oritur

$$\int x^\delta X''' dx = + \frac{x^{\delta-\alpha}\int x^\alpha X dx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} - \frac{\int x^\delta X dx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}$$
$$+ \frac{x^{\delta-\beta}\int x^\beta X dx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)} - \frac{\int x^\delta X dx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}$$
$$+ \frac{x^{\delta-\gamma}\int x^\gamma X dx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} - \frac{\int x^\delta X dx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)}$$

vbi ob rationes supra demonstratas tres postremi termini coalescunt in $+\frac{\int x^{\delta} X dx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}$, ita vt sit integrale quaesitum

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X dx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} + \frac{x^{-\beta}\int x^{\beta} X dx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}$$
$$+ \frac{x^{-\gamma}\int x^{\gamma} X dx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} + \frac{x^{-\delta}\int x^{\delta} X dx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}$$

siquidem omnes factores sint inter se inaequales. Casus autem quibus duo pluresue sunt aequales in corollariis explorabimus.

Coroll. 1.

1253. Si fuerint duo factores aequales nempe $\delta=\gamma$ seu si sit

$$P = N(\alpha+z)(\beta+z)(\gamma+z)^2$$

ex eadem forma pro X''' ante inuenta oritur integrale:

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X dx}{(\beta-\alpha)(\gamma-\alpha)^2} - \frac{x^{-\gamma}\int x^{\gamma} X dx}{(\beta-\alpha)(\gamma-\alpha)^2}$$
$$+ \frac{x^{-\beta}\int x^{\beta} X dx}{(\alpha-\beta)(\gamma-\beta)^2} - \frac{x^{-\gamma}\int x^{\gamma} X dx}{(\alpha-\beta)(\gamma-\beta)^2}$$
$$+ \frac{x^{-\gamma}\int \frac{dx}{x}\int x^{\gamma} X dx}{(\alpha-\gamma)(\beta-\gamma)} -$$

vbi

vbi membra negatiua ita repraesentari possunt

$$\frac{x^{-\gamma}\int x^{\gamma}X\,dx}{(\alpha-\beta)}\left(\frac{1}{(\gamma-\alpha)^2}-\frac{1}{(\gamma-\beta)^2}\right).$$

Coroll. 2.

1254. Si fuerint tres factores aequales vt sit

$$P=N(\alpha+z)(\beta+z)^3$$

ideoque $\delta=\gamma=\beta$ ex formula §. 1248. inuenta colligitur integrale

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}X\,dx}{(\beta-\alpha)^3}-\frac{x^{-\beta}\int x^{\beta}X\,dx}{(\beta-\alpha)^3}-\frac{x^{-\beta}\int\frac{dx}{x}\int x^{\beta}X\,dx}{(\beta-\alpha)^2}$$
$$+\frac{x^{-\beta}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}X\,dx}{\alpha-\beta}$$

Coroll. 3.

1255. Si omnes quatuor factores fuerint aequales, vt sit

$$P=N(\alpha+z)^4 \text{ existente } \delta=\gamma=\beta=\alpha$$

ex forma pro tribus aequalibus §. 1249. inuenta fit integrale

$$Nxy=x^{-\alpha}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\alpha}X\,dx.$$

Coroll. 4.

1256. Si habeatur $\beta=\alpha$ et $\delta=\gamma$, vt bini factores sint aequales scilicet $P=N(\alpha+z)^2(\gamma+z)^2$ ex §. 1247. vbi factores erant $(\alpha+z)^2(\gamma+z)$

colligitur integrale:

$$Nxy=\frac{x^{-\alpha}}{(\gamma-\alpha)^3}\int\frac{dx}{x}\int x^\alpha X dx-\frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int x^{\gamma-\alpha-1}dx\int x^\alpha X dx$$

$$-\frac{x^{-\alpha}}{(\gamma-\alpha)^3}\int x^\alpha X dx+\frac{x^{-\gamma}}{(\gamma-\alpha)^3}\int x^\gamma X dx+\frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^\gamma X dx$$

quae ob

$$\int x^{\gamma-\alpha-1}dx\int x^\alpha X dx=\frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int x^\alpha X dx-\frac{1}{\gamma-\alpha}\int x^\gamma X dx$$

contrahitur in hanc formam:

$$Nxy=\frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^\alpha X dx+\frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^\gamma X dx$$

$$-\frac{2x^{-\alpha}}{(\gamma-\alpha)^3}\int x^\alpha X dx-\frac{2x^{-\gamma}}{(\alpha-\gamma)^3}\int x^\gamma X dx.$$

Problema 170.

1257. Proposita hac aequatione differentiali quinti gradus

$$X=Ay+\frac{Bxdy}{dx}+\frac{Cx^2ddy}{dx^2}+\frac{Dx^3d^3y}{dx^3}+\frac{Ex^4d^4y}{dx^4}+\frac{Nx^5d^5y}{dx^5}$$

eius integrale per formulas integrales simplices euolvere.

Solutio.

Cum hic sit quantitas algebraica formanda:

$$P=A+B(z-1)+C(z-1)(z-2)+\ldots+N(z-1)(z-2)(z-3)(z-4)(z-5)$$

statua-

ſtatuatur

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z)(\varepsilon + z)$$

ac ſi hi factores omnes ſint inter ſe inaequales ex integrali praecedente nouam inſtituendo integrationem prodibit integrale quaeſitum:

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X dx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha)} + \frac{x^{-\beta}\int x^{\beta} X dx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta)}$$
$$+ \frac{x^{-\gamma}\int x^{\gamma} X dx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)(\varepsilon-\gamma)} + \frac{x^{-\delta}\int x^{\delta} X dx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)(\varepsilon-\delta)}$$
$$+ \frac{x^{-\varepsilon}\int x^{\varepsilon} X dx}{(\alpha-\varepsilon)(\beta-\varepsilon)(\gamma-\varepsilon)(\delta-\varepsilon)}$$

caſus quo duo pluresue factores ſunt aequales in Coroll. euoluemus.

Coroll. 1.

1258. Si fuerint duo factores aequales vt ſit $P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z)^2$ ideoque $\varepsilon = \delta$ ex praecedente problemate colligitur integrale

$$Nxy = + \frac{x^{-\alpha}\int x^{\alpha} X dx - x^{-\delta}\int x^{\delta} X dx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)^2}$$
$$+ \frac{x^{-\beta}\int x^{\beta} X dx - x^{-\delta}\int x^{\delta} X dx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)^2}$$
$$+ \frac{x^{-\gamma}\int x^{\gamma} X dx - x^{-\delta}\int x^{\delta} X dx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)^2}$$
$$+ \frac{x^{-\delta}\int \frac{dx}{x}\int x^{\delta} X dx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}.$$

Coroll. 2.

1259. Si fuerint tres factores aequales, vt sit $P = N(\alpha+z)(\beta+z)(\gamma+z)^3$ ideoque $\varepsilon=\delta=\gamma$ ex Coroll. 1. probl. praeced. colligitur:

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}X\,dx - x^{-\gamma}\int x^{\gamma}X\,dx}{(\beta-\alpha)(\gamma-\alpha)^2} - \frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}X\,dx}{(\beta-\alpha)(\gamma-\alpha)^2}$$

$$+ \frac{x^{-\beta}\int x^{\beta}X\,dx - x^{-\gamma}\int x^{\gamma}X\,dx}{(\alpha-\beta)(\gamma-\beta)^2} - \frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}X\,dx}{(\alpha-\beta)(\gamma-\beta)^2}$$

$$+ \frac{x^{-\gamma}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}X\,dx}{(\alpha-\gamma)(\beta-\gamma)}.$$

Coroll. 3.

1260. Si quatuor factores sint aequales, vt sit $P = N(\alpha+z)(\beta+z)^4$ ideoque $\varepsilon=\delta=\gamma=\beta$ erit per (1254)

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}X\,dx}{(\beta-\alpha)^4} - \frac{x^{-\beta}\int x^{\beta}X\,dx}{(\beta-\alpha)^4} - \frac{x^{-\beta}\int\frac{dx}{x}\int x^{\beta}X\,dx}{(\beta-\alpha)^3}$$

$$- \frac{x^{-\beta}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}X\,dx}{(\beta-\alpha)^2} + \frac{x^{-\beta}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}X\,dx}{\alpha-\beta}$$

ac si omnes quinque sint inter se aequales seu

$$P = N(\alpha+z)^5$$

erit integrale

$$Nxy = x^{-\alpha}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\alpha}X\,dx.$$

Coroll. 4.

Coroll. 4.

1261. Si P habeat duos factores quadratos, vt fit

$P = N(\alpha+z)(\beta+z)^2(\gamma+z)^2$ ideoque $\delta=\gamma$ et $\varepsilon=\beta$

erit ex §. 1253. integrale reductione necessaria facta

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}X\,dx - x^{-\beta}\int x^{\beta}X\,dx}{(\beta-\alpha)^2(\gamma-\alpha)^2} - \frac{x^{-\gamma}\int x^{\gamma}X\,dx + x^{-\beta}\int x^{\beta}X\,dx}{(\beta-\alpha)(\alpha-\gamma)^2(\beta-\gamma)}$$

$$+ \frac{x^{-\beta}\int\frac{dx}{x}\int x^{\beta}X\,dx}{(\alpha-\beta)(\gamma-\beta)^2} - \frac{x^{-\gamma}\int x^{\gamma}X\,dx + x^{-\beta}\int x^{\beta}X\,dx}{(\alpha-\beta)(\beta-\gamma)^3}$$

$$+ \frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}X\,dx}{(\alpha-\gamma)(\beta-\gamma)^2} - \frac{x^{-\gamma}\int x^{\gamma}X\,dx + x^{-\beta}\int x^{\beta}X\,dx}{(\alpha-\gamma)(\beta-\gamma)^3}$$

quae porro redigitur ad hanc formam

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}X\,dx}{(\beta-\alpha)^2(\gamma-\alpha)^2} + \frac{x^{-\beta}\int\frac{dx}{x}\int x^{\beta}X\,dx}{(\alpha-\beta)(\gamma-\beta)^2} - \frac{x^{-\beta}\int x^{\beta}X\,dx}{(\alpha-\beta)^2(\gamma-\beta)^2} - \frac{2x^{-\beta}\int x^{\beta}X\,dx}{(\alpha-\beta)(\gamma-\beta)^3}$$

$$+ \frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}X\,dx}{(\alpha-\gamma)(\beta-\gamma)^2} - \frac{x^{-\gamma}\int x^{\gamma}X\,dx}{(\alpha-\gamma)^2(\beta-\gamma)^2} - \frac{2x^{-\gamma}\int x^{\gamma}X\,dx}{(\alpha-\gamma)(\beta-\gamma)^3}.$$

Coroll. 5.

1262. Si P habeat et factorem quadratum et cubicum vt fit

$P = N(\alpha+z)^2(\gamma+z)^3$, ideoque $\beta=\alpha$ et $\varepsilon=\delta=\gamma$

ex §. 1254. colligitur integrale:

$$Nxy = \frac{x^{-\alpha}\int\frac{dx}{x}\int x^{\alpha}X\,dx}{(\gamma-\alpha)^3} - \frac{3x^{-\alpha}\int x^{\alpha}X\,dx}{(\gamma-\alpha)^4}$$

$$+ \frac{x^{-\gamma}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}X\,dx}{(\alpha-\gamma)^2} - \frac{2x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}X\,dx}{(\alpha-\gamma)^3} - \frac{3x^{-\gamma}\int x^{\gamma}X\,dx}{(\alpha-\gamma)^4}.$$

Scholion.

Scholion.

1263. Ex his formulis parum constat, quemadmodum eas vlterius pro maiori factorum numero continuari oporteat, si quidem factorum aliquot inter se fuerint aequales; integralium enim partes, quae factoribus inaequalibus respondent, legem seruant manifestam. Quae autem partibus aequalibus respondent, adhibita certa reductione commodius exprimi possunt. Veluti pro casu Coroll. 1. si breuitatis gratia ponatur $\alpha-\delta=p$, $\beta-\delta=q$ et $\gamma-\delta=r$ forma $x^{-\delta}\int x^{\delta}\mathrm{X}dx$ ducta est in:

$$\frac{1}{(p-q)(r-p)pp}+\frac{1}{(p-q)(q-r)qq}+\frac{1}{(r-p)(q-r)rr}$$

seu $\frac{(q-r)qqrr+(r-p)pprr+(p-q)ppqq}{(p-q)(q-r)(r-p)ppqqrr}$

cuius fractionis numerator est

$$-(p-q)(q-r)(r-p)(pq+pr+qr)$$

ita vt haec fractio reducatur ad istam

$$\frac{-pq-pr-qr}{ppqqrr}=-\frac{1}{pqr}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right).$$

Quando ergo est

$$\mathrm{P}=\mathrm{N}(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)^2$$

integrale ita se habet:

$$\mathrm{N}xy=\frac{x^{-\alpha}\int x^{\alpha}\mathrm{X}dx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)^2}+\frac{x^{-\beta}\int x^{\beta}\mathrm{X}dx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)^2}+\frac{x^{-\gamma}\int x^{\gamma}\mathrm{X}dx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)^2}$$
$$+\frac{x^{-\delta}\int\frac{dx}{x}\int x^{\delta}\mathrm{X}dx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}-\frac{x^{-\delta}\int x^{\delta}\mathrm{X}dx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\left(\frac{1}{\alpha-\delta}+\frac{1}{\beta-\delta}+\frac{1}{\gamma-\delta}\right).$$

Pro

Pro casu autem $P=N(\alpha+z)(\beta+z)(\gamma+z)^2$ habebitur

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}Xdx}{(\beta-\alpha)(\gamma-\alpha)^2}+\frac{x^{-\beta}\int x^{\beta}Xdx}{(\alpha-\beta)(\gamma-\beta)^2}$$
$$+\frac{x^{-\gamma}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}Xdx}{(\alpha-\gamma)(\beta-\gamma)}-\frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}Xdx}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{\alpha-\gamma}+\frac{1}{\beta-\gamma}\right)$$
$$+\frac{x^{-\gamma}\int x^{\gamma}Xdx}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{(\alpha-\gamma)^2}+\frac{1}{(\alpha-\gamma)(\beta-\gamma)}+\frac{1}{(\beta-\gamma)^2}\right).$$

Tum vero pro casu $P=N(\alpha+z)(\beta+z)^4$ fit

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}Xdx}{(\beta-\alpha)^4}+\frac{x^{-\beta}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx}{\alpha-\beta}$$
$$-\frac{x^{-\beta}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx}{\alpha-\beta}\cdot\frac{1}{\alpha-\beta}+\frac{x^{-\beta}\int\frac{dx}{x}\int x^{\beta}Xdx}{\alpha-\beta}\cdot\frac{1}{(\alpha-\beta)^2}-\frac{x^{-\beta}\int x^{\beta}Xdx}{\alpha-\beta}\cdot\frac{1}{(\alpha-\beta)^3}$$

At pro casu $P=(\alpha+z)(\beta+z)^2(\gamma+z)^2$ integrale ita se habet

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}Xdx}{(\beta-\alpha)^2(\gamma-\alpha)^2}+\frac{x^{-\beta}\int\frac{dx}{x}\int x^{\beta}Xdx}{(\alpha-\beta)(\gamma-\beta)^2}-\frac{x^{-\beta}\int x^{\beta}Xdx}{(\alpha-\beta)(\gamma-\beta)^2}\left(\frac{1}{\alpha-\beta}+\frac{2}{\gamma-\beta}\right)$$
$$+\frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}Xdx}{(\alpha-\gamma)(\beta-\gamma)^2}-\frac{x^{-\gamma}\int x^{\gamma}Xdx}{(\alpha-\gamma)(\beta-\gamma)^2}\left(\frac{1}{\alpha-\gamma}+\frac{2}{\beta-\gamma}\right).$$

Denique pro casu $P=N(\alpha+z)^2(\gamma+z)^2$ est

$$Nxy=\frac{x^{-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx}{(\gamma-\alpha)^3}-\frac{x^{-\alpha}\int x^{\alpha}Xdx}{(\gamma-\alpha)^3}\cdot\frac{3}{\gamma-\alpha}$$
$$+\frac{x^{-\gamma}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}Xdx}{(\alpha-\gamma)^2}-\frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}Xdx}{(\alpha-\gamma)^2}\cdot\frac{2}{\alpha-\gamma}+\frac{x^{-\gamma}\int x^{\gamma}Xdx}{(\alpha-\gamma)^2}\cdot\frac{3}{(\alpha-\gamma)^2}$$

vnde indoles harum formularum iam magis fit perſpicua, ſimulque patet partem integralis ex aliquot factoribus oriundam non pendere ab aequalitate reliquorum. Quocirca iam problema generale aggredi licebit.

Problema 171.

1264. Propoſita aequatione differentiali cuiuscunque gradus huius formae:

$$X = Ay + \frac{Bx\,dy}{dx} + \frac{Cx^2\,ddy}{dx^2} + \frac{Dx^3\,d^3y}{dx^3} + \dots + \frac{Nx^n\,d^ny}{dx^n}$$

ex qua forma algebraica hac lege formata

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \dots$$
$$\dots + N(z-1)(z-2)\dots\dots(z-n)$$

omnes factores habeat inter ſe inaequales; valorem ipſius y completum per formulas integrales ſimplices exhibere.

Solutio.

Sint primo formae P omnes factores ſimplices reales:

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\dots(\nu+z)$$

factorum numero exiſtente $= n$, et ex antecedentibus patet ex quolibet factore naſci integralis partem. Ad has partes inueniendas, eliciantur ſequentes valores:

1) poſito $z = -\alpha$ fit $\mathfrak{A} = \frac{P}{\alpha+z}$ ſeu $\mathfrak{A} = \frac{dP}{dz}$

2) poſito $z = -\beta$ fit $\mathfrak{B} = \frac{P}{\beta+z}$ ſeu $\mathfrak{B} = \frac{dP}{dz}$

3) poſito $z = -\gamma$ fit $\mathfrak{C} = \frac{P}{\gamma+z}$ ſeu $\mathfrak{C} = \frac{dP}{dz}$

etc.

Cum

Cum igitur ſit

$$(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\ldots(\nu-\alpha)=\frac{\mathfrak{A}}{N},$$

littera N ex ſuperioribus formis per diuiſionem tolletur, fietque integrale quaeſitum:

$$xy=\frac{1}{\mathfrak{A}}x^{-\alpha}\int x^{\alpha}X\,dx+\frac{1}{\mathfrak{B}}x^{-\beta}\int x^{\beta}X\,dx+\frac{1}{\mathfrak{C}}x^{-\gamma}\int x^{\gamma}X\,dx+\text{etc.}$$

quoad ſinguli factores fuerint exhauſti.

Quodſi iam forma P factores habeat imaginarios, partium inde ortarum imaginariarum ad realitatem reductio ſequenti modo inſtituetur. Quoniam bini factores ſimplices imaginarii praebent factorem duplicem realem, ponamus

$$(\alpha+z)(\beta+z)=ff+2fz\,\text{coſ.}\,\theta+zz$$

ita vt ſit

$\alpha=f(\text{coſ.}\,\theta+\sqrt{-1}.\,\text{ſin.}\,\theta)$ et $\beta=f(\text{coſ.}\,\theta-\sqrt{-1}.\,\text{ſin.}\,\theta)$

vnde primum valores literarum $\mathfrak{A}$ et $\mathfrak{B}$ definiantur, quarum cum vtraque deriuetur ex forma $\frac{dP}{dz}$, illa poſito $z=-\alpha$ haec vero poſito $z=-\beta$; in ipſa forma $\frac{dP}{dz}$ loco z vbique ſcribatur

$$-f(\text{coſ.}\,\theta\pm\sqrt{-1}.\,\text{ſin.}\,\theta),$$

prodeatque $\mathfrak{P}\pm\mathfrak{Q}\sqrt{-1}$; ac perſpicuum eſt fore

$$\mathfrak{A}=\mathfrak{P}+\mathfrak{Q}\sqrt{-1}\quad\text{et}\quad\mathfrak{B}=\mathfrak{P}-\mathfrak{Q}\sqrt{-1},$$

vbi notandum eſt, quantitates $\mathfrak{P}$ et $\mathfrak{Q}$ eſſe reales. Deinde cum ſit

$$x^{m+n\sqrt{-1}}=x^{m}e^{n\sqrt{-1}.\,lx}=x^{m}(\text{coſ.}(nlx)+\sqrt{-1}.\,\text{ſin.}(nlx))$$

si breuitatis ergo ponamus angulum $f\sin.\theta.lx = \Phi$ erit:

$$x^{\alpha} = x^{f\cos.\theta}(\cos.\Phi + \sqrt{-1}.\sin.\Phi);\ x^{-\alpha} = x^{-f\cos.\theta}(\cos.\Phi - \sqrt{-1}.\mathrm{si}.\Phi)$$

$$x^{\beta} = x^{f\cos.\theta}(\cos.\Phi - \sqrt{-1}.\sin.\Phi);\ x^{-\beta} = x^{-f\cos.\theta}(\mathrm{cs}.\Phi + \sqrt{-1}.\mathrm{si}.\Phi).$$

Quare pro binis partibus

$$\tfrac{1}{\mathfrak{A}}x^{-\alpha}\int x^{\alpha}X\,dx + \tfrac{1}{\mathfrak{B}}x^{-\beta}\int x^{\beta}X\,dx$$

ob $\mathfrak{A}\mathfrak{B} = \mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}$ habebimus:

$$+\frac{x^{-f\cos.\theta}}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}\left\{\begin{matrix}(\mathfrak{P}-\mathfrak{Q}\sqrt{-1})(\mathrm{cs}.\Phi-\sqrt{-1}.\mathrm{si}.\Phi)\int x^{f\cos.\theta}(\mathrm{cs}.\Phi+\sqrt{-1}.\mathrm{si}.\Phi)X\,dx\\(\mathfrak{P}+\mathfrak{Q}\sqrt{-1})(\mathrm{cs}.\Phi+\sqrt{-1}.\mathrm{si}.\Phi)\int x^{f\cos.\theta}(\mathrm{cs}.\Phi-\sqrt{-1}.\mathrm{si}.\Phi)X\,dx\end{matrix}\right\}$$

quae forma ob partes imaginarias se tollentes reducitur ad hanc:

$$\frac{2x^{-f\cos.\theta}(\mathfrak{P}\mathrm{cs}.\Phi-\mathfrak{Q}\mathrm{si}.\Phi)\int x^{f\cos.\theta}X\,dx\,\mathrm{cs}.\Phi+2x^{-f\cos.\theta}(\mathfrak{Q}\mathrm{cs}.\Phi+\mathfrak{P}\mathrm{si}.\Phi)\int x^{f\cos.\theta}X\,dx\,\mathrm{si}.\Phi}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}.$$

Talisque forma ad integrale accedit, quoties forma P huiusmodi habet factorem duplicem $ff + 2fz\cos.\theta + zz$.

Coroll. 1.

1265. Etsi autem factorum simplicium ipsius P quidam sunt imaginarii, eorum qui sunt reales euolutio inde non perturbatur, sed ex singulis partes in integrale inferendae a natura reliquorum factorum minime pendent.

Coroll. 2.

1266. Pars integralis ex binis factoribus imaginariis seu vno factore duplici oriunda aliquanto succinctius

cinctius repraesentari potest si ponatur

$$\mathfrak{P} = \mathfrak{O} \operatorname{cos.} \zeta \quad \text{et} \quad \mathfrak{Q} = \mathfrak{O} \operatorname{sin.} \zeta ,$$

sic enim ea fiet:

$$\tfrac{2}{\mathfrak{D}} x^{-f\cos.\theta} (\operatorname{cos.}(\zeta+\Phi) \int x^{f\cos.\theta} X dx \operatorname{cs.} \Phi + \operatorname{si.}(\zeta+\Phi) \int x^{f\cos.\theta} X dx \operatorname{si.} \Phi)$$

vbi ζ et θ sunt anguli constantes Φ vero variabilis ob $\Phi = f \operatorname{sin.} \theta . l x$.

Problema 172.

1267. Si pro aequatione differentiali in praec. probl. proposita quantitas algebraica P inde formata duos habeat factores simplices aequales, integralis partem inde oriundam inuestigare.

Solutio.

In forma ergo ante exhibita:

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z) \text{ etc.}$$

ponamus esse $\beta = \alpha$, quoniam vero tum vtraque integralis pars oritur infinita, altera signo +, altera signo − affecta, ita vt iunctim sumtae partem constituant finitam, ad hanc eliciendam statuamus $\beta = \alpha - \omega$ denotante ω quantitatem euanescentem, eritque

$$\mathfrak{A} = -N\omega(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha) \text{ etc. et}$$

$$\mathfrak{B} = +N\omega(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta) \text{ etc.}$$

Ponatur iam

$$\frac{P}{(\alpha+z)(\beta+z)} = \frac{P}{(\alpha+z)^2} = Q$$

vt fit

$$Q = N(\gamma + z)(\delta + z)(\epsilon + z) \text{ etc.}$$

ac manifeſtum eſt fieri

$\mathfrak{A} = -\omega Q$ poſito $z = -\alpha$ et

$\mathfrak{B} = \omega Q$ poſito $z = -\beta = -\alpha + \omega$

vnde intelligitur valorem ipſius Q poſteriorem excedere priorem ſuo differentiali dQ ſi fiat

$$z = -\alpha \quad \text{et} \quad dz = \omega,$$

ita vt ſit

$$\mathfrak{B} = \omega\left(Q + \omega.\frac{dQ}{dz}\right) \text{ poſito } z = -\alpha,$$

hincque

$$\frac{1}{\mathfrak{B}} = \frac{1}{\omega Q} - \frac{dQ}{QQdz} = \frac{1}{\omega Q} + \frac{1}{dz}d.\frac{1}{Q}, \text{ exiſtente } \frac{1}{\mathfrak{A}} = -\frac{1}{\omega Q}$$

tum vero cum ſit

$$x^{\beta} = x^{\alpha}x^{-\omega} = x^{\alpha}(1 - \omega lx) \text{ et } x^{-\beta} = x^{-\alpha}(1 + \omega lx),$$

binae partes integralis quaeſitae erunt

$$-\frac{1}{\omega Q}x^{-\alpha}\int x^{\alpha}Xdx + \left(\frac{1}{\omega Q} + \frac{1}{dz}d.\frac{1}{Q}\right)x^{-\alpha}(1 + \omega lx)\int x^{\alpha}Xdx(1 - \omega lx)$$

vbi cum membra per ω diuiſa ſe deſtruant, reſultat

$$\frac{1}{Q}x^{-\alpha}(lx\int x^{\alpha}Xdx - \int x^{\alpha}Xdxlx) + \frac{1}{dz}d.\frac{1}{Q}.x^{-\alpha}\int x^{\alpha}Xdx$$

ſeu $\frac{1}{Q}x^{-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx + \frac{1}{dz}d.\frac{1}{Q}.x^{-\alpha}\int x^{\alpha}Xdx$

ſiquidem tam in valore $\frac{1}{Q}$ quam in $\frac{1}{dz}d.\frac{1}{Q}$ vbique loco z ſcribatur $-\alpha$. Cum vero ſit $Q = \frac{P}{(\alpha + z)^2}$, hi valores inde facile inueniuntur.

Coroll. 1.

Coroll. 1.

1268. Quodsi ergo quantitas algebraica P ex aequatione differentiali formata factorem habeat quadratum $(\alpha+z)^2$, inde in integrale transferenda est haec portio

$$\frac{(\alpha+z)^2}{P}x^{-\alpha}\int\frac{dx}{x}\int x^\alpha X\,dx+\frac{1}{dz}d.\frac{(\alpha+z)^2}{P}x^{-\alpha}\int x^\alpha X\,dx$$

posito $z=-\alpha$; dum si hic factor $\alpha+z$ esset solitarius integralis pars inde oriunda foret:

$$\frac{\alpha+z}{P}x^{-\alpha}\int x^\alpha X\,dx \text{ posito } z=-\alpha.$$

Coroll. 2.

1269. Cum sit $Q=\frac{P}{(\alpha+z)^2}$, casu $z=-\alpha$ fiet $Q=\frac{ddP}{2dz^2}$ verum quia hic ipsi z iam valor determinatus est tributus hinc $\frac{dQ}{dz}$ colligere non licet, sed prima est vtendum qua fit $\frac{dQ}{dz}=\frac{(\alpha+z)dP-2Pdz}{(\alpha+z)^3dz}$, cuius fractionis cum numerator et denominator casu $z=-\alpha$ euanescat, erit pro eodem casu

$$\frac{dQ}{dz}=\frac{(\alpha+z)ddP-dzdP}{3(\alpha+z)^2dz^2}=\frac{(\alpha+z)d^3P}{6(\alpha+z)dz^3}=\frac{d^3P}{6dz^3}.$$

Coroll. 3.

1270. Hoc valore inuento, quia est eodem casu $z=-\alpha$ quantitas $Q=\frac{ddP}{2dz^2}$ erit

$$\frac{1}{dz}d.\frac{1}{Q}=-\frac{dQ}{QQdz}=-\frac{2dzd^3P}{3ddP^2} \text{ seu}$$

$$\frac{2}{dz}d.\frac{1}{Q}=\frac{2dz}{3}d.\frac{1}{ddP},$$

ex

ex quibus formulis, si factores ipsius P non sint euoluti partes integralis facilius reperiuntur.

Problema 173.

1271. Si pro aequatione differentiali praecedente quantitas algebraica P inde formata factorem habeat cubicum $(\alpha+z)^3$ integralis partem inde oriundam inuestigare.

Solutio.

Ponamus ergo esse

$$P=(\alpha+z)^2(\gamma+z)R \text{ existente } \gamma=\alpha-\omega,$$

vbi ω pro quantitate euanescente assumitur. Quod ergo ante erat Q id hinc fit $Q=(\gamma+z)R$, et facto $z=-\alpha$ erit $Q=-\omega R$ si etiam in R ponatur $z=-\alpha$. Deinde cum sit

$$\frac{dQ}{dz}=R+\frac{(\gamma+z)dR}{dz}=R-\frac{\omega dR}{dz}$$

eodem casu erit

$$\frac{1}{dz}d.\frac{1}{Q}=-\frac{1}{\omega\omega R}+\frac{dR}{\omega RRdz}=-\frac{1}{\omega\omega}\cdot\frac{1}{R}-\frac{1}{\omega dz}d.\frac{1}{R}.$$

Quocirca ex factore quadrato $(\alpha+z)^2$ per praecedens problema haec obtinetur integralis pars:

$$-\frac{1}{\omega R}x^{-\alpha}\int\frac{dx}{x}\int x^\alpha X dx-\left(\frac{1}{\omega\omega R}+\frac{1}{\omega dz}d.\frac{1}{R}\right)x^{-\alpha}\int x^\alpha X dx$$

cuius ambo membra in infinitum excrescunt ob $\omega=0$. Adiiciamus autem partem ex tertio factore

$$\gamma+z=\alpha-\omega+z$$

oriundam

oriundam, quae ob $\frac{P}{\gamma+z} = (\alpha + z)^\epsilon R$ eſt

$$\frac{1}{(\alpha+z)^\epsilon R} x^{-\gamma} \int x^\gamma X\, dx \text{ poſito } z = -\gamma = -\alpha + \omega.$$

Quod ſi iam R vt ante is fuerit valor, qui oritur poſito $z = -\alpha$; augendo hunc valorem particula ω, loco $\frac{1}{R}$ ſcribi debet

$$\frac{1}{R} + \frac{\omega}{dz} d.\frac{1}{R} + \frac{\omega^2}{1.2\, dz^2} dd.\frac{1}{R} \text{ etc.}$$

ſi quidem valorem $z = -\alpha$ et hic retineamus; vnde haec integralis pars ob $\alpha + z = \omega$ erit

$$\left(\frac{1}{\omega\omega R} + \frac{1}{\omega\, dz} d.\frac{1}{R} + \frac{1}{2\, dz^2} dd.\frac{1}{R}\right) x^{-\alpha+\omega} \int x^{\alpha-\omega} X\, dx$$

ſicque manifeſtum eſt illum ipſius $\frac{1}{R}$ valorem vsque ad ſecundam poteſtatem ipſius ω continuari debuiſſe, atque eadem lege hic alteram partem x inuoluentem exprimi conueniet. Ad quod obſeruo ſi habeatur huiusmodi formula $x^\omega \int x^{-\omega} V\, dx$ ſecundum poteſtates ipſius ω euoluenda, id hac ratione commodiſſime fieri. Poſito

$$v = x^\omega \int x^{-\omega} V\, dx,$$

vt ſit

$$x^{-\omega} v = \int x^{-\omega} V\, dx$$

erit differentiando

$$dv - \frac{\omega v\, dx}{x} = V\, dx,$$

quare poſito

$$v = T + \omega T' + \omega^2 T'' + \omega^3 T''' \text{ etc}$$

 habe-

habebitur terminos ſecundum poteſtates ipſius ω diſponendo:

$$\left.\begin{aligned} &dT + \omega dT' + \omega\omega dT'' + \omega^3 dT''' + \text{etc.} \\ &-Vdx - \omega T\tfrac{dx}{x} - \omega\omega T'\tfrac{dx}{x} - \omega^3 T''\tfrac{dx}{x} - \text{etc.}\end{aligned}\right\} = 0$$

ideoque

$$T = \int V dx;\; T' = \int \tfrac{dx}{x}\int V dx;\; T'' = \int \tfrac{dx}{x}\int\tfrac{dx}{x}\int V dx \text{ etc.}$$

Conſequenter cum in applicatione ſit $V = x^\alpha X$ erit pars integralis ex factore $\gamma + z = \alpha - \omega + z$ nata:

$$\left(\tfrac{1}{\omega\omega R} + \tfrac{1}{\omega dz}d.\tfrac{1}{R} + \tfrac{1}{2dz^2}dd.\tfrac{1}{R}\right)x^{-\alpha}\left(\int x^\alpha X dx + \omega\int\tfrac{dx}{x}\int x^\alpha X dx + \omega^2\int\tfrac{dx}{x}\int\tfrac{dx}{x}\int x^\alpha X dx\right)$$

qua cum parte ex $(\alpha + z)^2$ nata iunctim ſumta omnia membra infinita ſe mutuo deſtruunt, et pro quantitatis $P = (\alpha + z)^3 R$ factore cubico $(\alpha + z)^3$ in integrale ingreditur haec pars:

$$\tfrac{1}{R}x^{-\alpha}\int\tfrac{dx}{x}\int\tfrac{dx}{x}\int x^\alpha X dx + \tfrac{1}{dz}d.\tfrac{1}{R}x^{-\alpha}\int\tfrac{dx}{x}\int x^\alpha X dx + \tfrac{1}{2dz^2}dd.\tfrac{1}{R}x^{-\alpha}\int x^\alpha X dx$$

ſi modo in quantitate $R = \frac{P}{(\alpha + z)^3}$ vbique ſcribatur $z = -\alpha$.

Coroll. 1.

1272. Methodus in ſolutione huius problematis adhibita facile ad quotcunque factores aequales extendi poteſt. Si enim fuerit $(\alpha + z)^m$ factor quantitatis P, atque in hac fractione $\frac{(\alpha + z)^m}{P}$ fuis-

que

que differentialibus, postquam fuerint euoluta ponatur $z = -\alpha$, partes integralis inde natae ita se habebunt

Factor quant. P	$\alpha + z$	$(\alpha + z)^2$	$(\alpha + z)^3$
Pars integralis	$\frac{\alpha+z}{P} x^{-\alpha} \int x^{\alpha} X dx$	$\frac{(\alpha+z)^2}{P} x^{-\alpha} \int \frac{dx}{x} \int x^{\alpha} X dx$	$\frac{(\alpha+z)^3}{P} x^{-\alpha} \int \frac{dx}{x} \int \frac{dx}{x} \int x^{\alpha} X dx$
		$\frac{1}{dz} d. \frac{(\alpha+z)^2}{P} x^{-\alpha} \int x^{\alpha} X dx$	$\frac{1}{dz} d. \frac{(\alpha+\alpha)^3}{P} x^{-\alpha} \int \frac{dx}{x} \int x^{\alpha} X dx$
			$\frac{1}{2dz^2} dd. \frac{(\alpha+z)^3}{P} x^{-\alpha} \int x^{\alpha} X dx.$

Coroll. 2.

1273. Si fuerint duo pluresue factores duplices inter se aequales sumtis

$\alpha = f(\text{cos.}\,\theta + \sqrt{-1}.\,\text{sin.}\,\theta)$ et $\beta = f(\text{cos.}\,\theta - \sqrt{-1}.\,\text{sin.}\,\theta)$, partes pro $(\alpha+z)^2$ et $(\beta+z)^2$ seorsim euolutae methodo supra adhibita non difficulter coniungentur et ad realitatem reducentur.

Scholion.

1274. Simili methodo, qua hoc caput est pertractatum, in euolutione Capitis III. huius sectionis vti oportebat, neque tum vllum periculum in errores prolabendi fuisset pertimescendum. Superfluum autem nunc foret, errores ibi commissos hic emendare, cum non solum methodus plane esset eadem, sed etiam aequatio hic tractata facile in formam ibi consideratam transmutari queat et vicissim.

Quodſi enim in aequatione Cap. III.

$$X = Ay + \frac{B\,dy}{dx} + \frac{C\,ddy}{dx^2} + \frac{D\,d^3y}{dx^3} + \frac{E\,d^4y}{dx^4} + \text{etc.}$$

ſtatuatur $x = lv$, vt ſit $dx = \frac{dv}{v}$, functio autem X abeat in functionem ipſius v quae ſit V, proueniet aequatio eius formae quam hic tractauimus. Dum autem ibi elementum dx pro conſtanti eſt habitum, ad hanc conditionem exuendam ponamus

$$dy = p\,dx,\ dp = q\,dx,\ dq = r\,dx,\ dr = s\,dx \text{ etc.}$$

vt haec aequatio reſultet:

$$X = V = Ay + Bp + Cq + Dr + Es + \text{etc.}$$

Nunc autem ob $dx = \frac{dv}{v}$, adipiſcimur elemento dv conſtante ſumto:

$$p = \frac{dy}{dx} = \frac{v\,dy}{dv}$$

$$q = \frac{dp}{dx} = \frac{vv\,ddy}{dv^2} + \frac{v\,dy}{dv}$$

$$r = \frac{dq}{dx} = \frac{v^3\,d^3y}{dv^3} + \frac{3v^2\,ddy}{dv^2} + \frac{v\,dy}{dv}$$

$$s = \frac{dr}{dx} = \frac{v^4\,d^4y}{dv^4} + \frac{6v^3\,d^3y}{dv^3} + \frac{7vv\,ddy}{dv^2} + \frac{v\,dy}{dv}$$

$$t = \frac{ds}{dx} = \frac{v^5\,d^5y}{dv^5} + \frac{10v^4\,d^4y}{dv^4} + \frac{25v^3\,d^3y}{dv^3} + \frac{15vv\,ddy}{dv^2} + \frac{v\,dy}{dv}.$$

Quare

Quare aequatio inter v et y erit haec

$$\begin{array}{llllll} V = Ay + & \frac{Bvdy}{dv} + & \frac{Cvvddy}{dv^2} + & \frac{Dv^3d^3y}{dv^3} + & \frac{Ev^4d^4y}{dv^4} + & \frac{Fv^5d^5y}{dv^5} \\ & & +C & +3D & +6E & +10F \\ & & & +D & +7E & +25F \\ & & & & +E & +15F \\ & & & & & +F \end{array}$$

cuius integrationem hic docuimus. Imprimis autem notandum eſt quantitatem algebraicam P hinc formandam

$$\begin{aligned} P = A &+ (B + C + D + E + F)(z-1) \\ &+ (C + 3D + 7E + 15F)(z-1)(z-2) \\ &+ (D + 6E + 25F)(z-1)(z-2)(z-3) + \text{etc.} \end{aligned}$$

ad hanc formam reduci:

$$P = A + B(z-1) + C(z-1)^2 + D(z-1)^3 + E(z-1)^4 + F(z-1)^5 + \text{etc.}$$

quae quantitas algebraica ab illa, qua in Cap. III. ad integrationem ſumus vſi, hoc tantum differt, quod ibi littera z id quod hic formula $z-1$ expreſſimus; ex quo etiam ambarum integratio facillime altera ad alteram reducitur.

Quodſi enim in aequatione Cap. III.

$$X = Ay + \frac{B\,dy}{dx} + \frac{C\,ddy}{dx^2} + \frac{D\,d^3y}{dx^3} + \frac{E\,d^4y}{dx^4} + \text{etc.}$$

ſtatuatur $x = lv$, vt ſit $dx = \frac{dv}{v}$, functio autem X abeat in functionem ipſius v quae ſit V, proueniet aequatio eius formae quam hic tractauimus. Dum autem ibi elementum dx pro conſtanti eſt habitum, ad hanc conditionem exuendam ponamus

$$dy = p\,dx,\ dp = q\,dx,\ dq = r\,dx,\ dr = s\,dx \text{ etc.}$$

vt haec aequatio reſultet:

$$X = V = Ay + Bp + Cq + Dr + Es + \text{etc.}$$

Nunc autem ob $dx = \frac{dv}{v}$, adipiſcimur elemento dv conſtante ſumto:

$$p = \frac{dy}{dx} = \frac{v\,dy}{dv}$$

$$q = \frac{dp}{dx} = \frac{vv\,ddy}{dv^2} + \frac{v\,dy}{dv}$$

$$r = \frac{dq}{dx} = \frac{v^3\,d^3y}{dv^3} + \frac{3v^2\,ddy}{dv^2} + \frac{v\,dy}{dv}$$

$$s = \frac{dr}{dx} = \frac{v^4\,d^4y}{dv^4} + \frac{6v^3\,d^3y}{dv^3} + \frac{7vv\,ddy}{dv^2} + \frac{v\,dy}{dv}$$

$$t = \frac{ds}{dx} = \frac{v^5\,d^5y}{dv^5} + \frac{10v^4\,d^4y}{dv^4} + \frac{25v^3\,d^3y}{dv^3} + \frac{15vv\,ddy}{dv^2} + v\,d$$

Quare aequatio inter v et y erit haec

$$V = Ay + \frac{Bvdy}{dv} + \frac{Cvvddy}{dv^2} + \frac{Dv^3d^3y}{dv^3} + \frac{Ev^4d^4y}{dv^4} + \frac{Fv^5d^5y}{dv^5}$$
$$+C \quad +3D \quad +6E \quad +10F$$
$$+D \quad +7E \quad +25F$$
$$+E \quad +15F$$
$$+F$$

cuius integrationem hic docuimus. Imprimis autem notandum est quantitatem algebraicam P hinc formandam

$$P = A + (B+C+D+E+F)(z-1)$$
$$+(C+3D+7E+15F)(z-1)(z-2)$$
$$+(D+6E+25F)(z-1)(z-2)(z-3) + \text{etc.}$$

ad hanc formam reduci:

$$P = A + B(z-1) + C(z-1)^2 + D(z-1)^3 + E(z-1)^4$$
$$+ F(z-1)^5 \ldots$$

quae quantitas algebraica ab illa, ... hoc exem- ad integrationem sumus ... quod ibi littera z id ... pressimus; ex quo et ... lime altera ad altera...

$\frac{1+uu}{-1)}$

$\frac{-\sqrt{(1+uu)}}{uu+1)}$

$\frac{\ldots + bb)\cos.\Phi)}{\ldots + ab.\cos.\Phi}$

$\frac{-(aa+bb)\cos.\Phi}{bb+2ab\cos.\Phi}$.

$l\frac{x}{a}$.

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vnde indoles harum formularum iam magis fit perspicua, simulque patet partem integralis ex aliquot factoribus oriundam non pendere ab aequalitate reliquorum. Quocirca iam problema generale aggredi licebit.

Problema 171.

1264. Proposita aequatione differentiali cuiuscunque gradus huius formae:

$$X = Ay + \frac{Bx\,dy}{dx} + \frac{Cx^2\,ddy}{dx^2} + \frac{Dx^3\,d^3y}{dx^3} + \ldots + \frac{Nx^n\,d^ny}{dx^n}$$

ex qua forma algebraica hac lege formata

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \ldots$$
$$\ldots + N(z-1)(z-2)\ldots\ldots(z-n)$$

omnes factores habeat inter se inaequales; valorem ipsius y completum per formulas integrales simplices exhibere.

Solutio.

Sint primo formae P omnes factores simplices reales:

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\ \ldots\ldots\ (\nu+z)$$

factorum numero existente $=n$, et ex antecedentibus patet ex quolibet factore nasci integralis partem. Ad has partes inueniendas, eliciantur sequentes valores:

1) posito $z=-\alpha$ fit $\mathfrak{A} = \frac{P}{\alpha+z}$ seu $\mathfrak{A} = \frac{dP}{dz}$

2) posito $z=-\beta$ fit $\mathfrak{B} = \frac{P}{\beta+z}$ seu $\mathfrak{B} = \frac{dP}{dz}$

3) posito $z=-\gamma$ fit $\mathfrak{C} = \frac{P}{\gamma+z}$ seu $\mathfrak{C} = \frac{dP}{dz}$

etc.

Cum

Cum igitur sit

$$(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\ldots\ldots(\nu-\alpha)=\frac{\mathfrak{A}}{N},$$

littera N ex superioribus formis per diuisionem tolletur, fietque integrale quaesitum:

$$xy=\tfrac{1}{\mathfrak{A}}x^{-\alpha}\int x^{\alpha}X\,dx+\tfrac{1}{\mathfrak{B}}x^{-\beta}\int x^{\beta}X\,dx+\tfrac{1}{\mathfrak{C}}x^{-\gamma}\int x^{\gamma}X\,dx+\text{etc.}$$

quoad singuli factores fuerint exhausti.

Quodsi iam forma P factores habeat imaginarios, partium inde ortarum imaginariarum ad realitatem reductio sequenti modo instituetur. Quoniam bini factores simplices imaginarii praebent factorem duplicem realem, ponamus

$$(\alpha+z)(\beta+z)=ff+2fz\cos.\theta+zz$$

ita vt sit

$$\alpha=f(\cos.\theta+\sqrt{-1}.\sin.\theta) \text{ et } \beta=f(\cos.\theta-\sqrt{-1}.\sin.\theta)$$

vnde primum valores literarum $\mathfrak{A}$ et $\mathfrak{B}$ definiantur, quarum cum vtraque deriuetur ex forma $\frac{dP}{dz}$, illa posito $z=-\alpha$ haec vero posito $z=-\beta$; in ipsa forma $\frac{dP}{dz}$ loco z vbique scribatur

$$-f(\cos.\theta\pm\sqrt{-1}.\sin.\theta),$$

prodeatque $\mathfrak{P}\pm\mathfrak{Q}\sqrt{-1}$; ac perspicuum est fore

$$\mathfrak{A}=\mathfrak{P}+\mathfrak{Q}\sqrt{-1} \quad\text{et}\quad \mathfrak{B}=\mathfrak{P}-\mathfrak{Q}\sqrt{-1},$$

vbi notandum est, quantitates $\mathfrak{P}$ et $\mathfrak{Q}$ esse reales. Deinde cum sit

$$x^{m+n\sqrt{-1}}=x^{m}e^{n\sqrt{-1}.lx}=x^{m}(\cos.(nlx)+\sqrt{-1}.\sin.(nlx))$$

si breuitatis ergo ponamus angulum $f\sin.\theta.lx = \Phi$ erit:

$$x^{\alpha} = x^{f\cos.\theta}(\cos.\Phi + \sqrt{-1}.\sin.\Phi);\; x^{-\alpha} = x^{-f\cos.\theta}(\cos.\Phi - \sqrt{-1}.\sin.\Phi)$$

$$x^{\beta} = x^{f\cos.\theta}(\cos.\Phi - \sqrt{-1}.\sin.\Phi);\; x^{-\beta} = x^{-f\cos.\theta}(\cos.\Phi + \sqrt{-1}.\sin.\Phi).$$

Quare pro binis partibus

$$\tfrac{1}{\mathfrak{A}} x^{-\alpha} \int x^{\alpha} X\, dx + \tfrac{1}{\mathfrak{B}} x^{-\beta} \int x^{\beta} X\, dx$$

ob $\mathfrak{A}\mathfrak{B} = \mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}$ habebimus:

$$+\frac{x^{-f\cos.\theta}}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}\left\{\begin{matrix}(\mathfrak{P}-\mathfrak{Q}\sqrt{-1})(\cos.\Phi-\sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta}(\cos.\Phi+\sqrt{-1}.\sin.\Phi)X dx\\ (\mathfrak{P}+\mathfrak{Q}\sqrt{-1})(\cos.\Phi+\sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta}(\cos.\Phi-\sqrt{-1}.\sin.\Phi)X dx\end{matrix}\right\}$$

quae forma ob partes imaginarias se tollentes reducitur ad hanc:

$$\frac{2x^{-f\cos.\theta}(\mathfrak{P}\cos.\Phi-\mathfrak{Q}\sin.\Phi)\int x^{f\cos.\theta}X dx \cos.\Phi + 2x^{-f\cos.\theta}(\mathfrak{Q}\cos.\Phi+\mathfrak{P}\sin.\Phi)\int x^{f\cos.\theta}X dx \sin.\Phi}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}.$$

Talisque forma ad integrale accedit, quoties forma P huiusmodi habet factorem duplicem $ff + 2fz\cos.\theta + zz$.

Coroll. 1.

1265. Etsi autem factorum simplicium ipsius P quidam sunt imaginarii, eorum qui sunt reales euolutio inde non perturbatur, sed ex singulis partes in integrale inferendae a natura reliquorum factorum minime pendent.

Coroll. 2.

1266. Pars integralis ex binis factoribus imaginariis seu vno factore duplici oriunda aliquanto succinctius

cinctius repraesentari potest si ponatur

$$\mathfrak{P} = \mathfrak{D} \operatorname{cos.} \zeta \quad \text{et} \quad \mathfrak{Q} = \mathfrak{D} \operatorname{sin.} \zeta ,$$

sic enim ea fiet:

$$\tfrac{1}{\mathfrak{D}} x^{-f\cos.\theta} (\operatorname{cos.}(\zeta+\Phi) \int x^{f\cos.\theta} X dx \operatorname{cs.} \Phi + \operatorname{si.}(\zeta+\Phi) \int x^{f\cos.\theta} X dx \operatorname{si.} \Phi)$$

vbi ζ et θ sunt anguli constantes Φ vero variabilis ob $\Phi = f \operatorname{sin.} \theta . l x$.

Problema 172.

1267. Si pro aequatione differentiali in praec. probl. proposita quantitas algebraica P inde formata duos habeat factores simplices aequales, integralis partem inde oriundam inuestigare.

Solutio.

In forma ergo ante exhibita:

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z) \text{ etc.}$$

ponamus esse $\beta = \alpha$, quoniam vero tum vtraque integralis pars oritur infinita, altera signo +, altera signo − affecta, ita vt iunctim sumtae partem constituant finitam, ad hanc eliciendam statuamus $\beta = \alpha - \omega$ denotante ω quantitatem euanescentem, eritque

$$\mathfrak{A} = -N\omega(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha) \text{ etc. et}$$
$$\mathfrak{B} = +N\omega(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta) \text{ etc.}$$

Ponatur iam

$$\frac{P}{(\alpha+z)(\beta+z)} = \frac{P}{(\alpha+z)^2} = Q$$

vt fit

$$Q = N(\gamma + z)(\delta + z)(\varepsilon + z) \text{ etc.}$$

ac manifeſtum eſt fieri

$$\mathfrak{A} = -\omega Q \text{ poſito } z = -\alpha \text{ et}$$

$$\mathfrak{B} = \omega Q \text{ poſito } z = -\beta = -\alpha + \omega$$

vnde intelligitur valorem ipſius Q poſteriorem excedere priorem ſuo differentiali dQ ſi fiat

$$z = -\alpha \text{ et } dz = \omega,$$

ita vt fit

$$\mathfrak{B} = \omega(Q + \omega . \tfrac{dQ}{dz}) \text{ poſito } z = -\alpha,$$

hincque

$$\tfrac{1}{\mathfrak{B}} = \tfrac{1}{\omega Q} - \tfrac{dQ}{QQdz} = \tfrac{1}{\omega Q} + \tfrac{1}{dz} d.\tfrac{1}{Q}, \text{ exiſtente } \tfrac{1}{\mathfrak{A}} = -\tfrac{1}{\omega Q}$$

tum vero cum fit

$$x^{\beta} = x^{\alpha} x^{-\omega} = x^{\alpha}(1 - \omega l x) \text{ et } x^{-\beta} = x^{-\alpha}(1 + \omega l x),$$

binae partes integralis quaeſitae erunt

$$-\tfrac{1}{\omega Q} x^{-\alpha} \int x^{\alpha} X dx + (\tfrac{1}{\omega Q} + \tfrac{1}{dz} d.\tfrac{1}{Q}) x^{-\alpha}(1 + \omega l x) \int x^{\alpha} X dx (1 - \omega l x)$$

vbi cum membra per ω diuiſa ſe deſtruant, reſultat

$$\tfrac{1}{Q} x^{-\alpha}(lx \int x^{\alpha} X dx - \int x^{\alpha} X dx\, lx) + \tfrac{1}{dz} d.\tfrac{1}{Q} . x^{-\alpha} \int x^{\alpha} X dx$$

ſeu $\tfrac{1}{Q} x^{-\alpha} \int \tfrac{dx}{x} \int x^{\alpha} X dx + \tfrac{1}{dz} d.\tfrac{1}{Q} . x^{-\alpha} \int x^{\alpha} X dx$

ſiquidem tam in valore $\tfrac{1}{Q}$ quam in $\tfrac{1}{dz} d.\tfrac{1}{Q}$ vbique loco z ſcribatur $-\alpha$. Cum vero ſit $Q = \tfrac{P}{(\alpha + z)^2}$, hi valores inde facile inueniuntur.

Coroll. 1.

Coroll. 1.

1268. Quodſi ergo quantitas algebraica P ex aequatione differentiali formata factorem habeat quadratum $(\alpha+z)^2$, inde in integrale transferenda eſt haec portio

$$\frac{(\alpha+z)^2}{P} x^{-\alpha} \int \frac{dx}{x} \int x^{\alpha} X dx + \frac{1}{dz} d. \frac{(\alpha+z)^2}{P} x^{-\alpha} \int x^{\alpha} X dx$$

poſito $z = -\alpha$; dum ſi hic factor $\alpha+z$ eſſet ſolitarius integralis pars inde oriunda foret:

$$\frac{\alpha+z}{P} x^{-\alpha} \int x^{\alpha} X dx \text{ poſito } z = -\alpha.$$

Coroll. 2.

1269. Cum ſit $Q = \frac{P}{(\alpha+z)^2}$, caſu $z = -\alpha$ fiet $Q = \frac{ddP}{2dz^2}$ verum quia hic ipſi z iam valor determinatus eſt tributus hinc $\frac{dQ}{dz}$ colligere non licet, ſed prima eſt vtendum qua fit $\frac{dQ}{dz} = \frac{(\alpha+z)dP - 2Pdz}{(\alpha+z)^3 dz}$, cuius fractionis cum numerator et denominator caſu $z = -\alpha$ euaneſcat, erit pro eodem caſu

$$\frac{dQ}{dz} = \frac{(\alpha+z)ddP - dzdP}{3(\alpha+z)^2 dz^2} = \frac{(\alpha+z)d^3P}{6(\alpha+z)dz^3} = \frac{d^3P}{6dz^3}.$$

Coroll. 3.

1270. Hoc valore inuento, quia eſt eodem caſu $z = -\alpha$ quantitas $Q = \frac{ddP}{2dz^2}$ erit

$$\frac{1}{dz} d. \frac{1}{Q} = -\frac{dQ}{QQdz} = -\frac{2dzd^3P}{3ddP^2} \text{ ſeu}$$

$$\frac{1}{dz} d. \frac{1}{Q} = \frac{2dz}{3} d. \frac{1}{ddP},$$

ex quibus formulis, si factores ipsius P non sint euoluti partes integralis facilius reperiuntur.

Problema 173.

1271. Si pro aequatione differentiali praecedente quantitas algebraica P inde formata factorem habeat cubicum $(\alpha+z)^3$ integralis partem inde oriundam inuestigare.

Solutio.

Ponamus ergo esse

$$P=(\alpha+z)^2(\gamma+z)R \text{ existente } \gamma=\alpha-\omega,$$

vbi ω pro quantitate euanescente assumitur. Quod ergo ante erat Q id hinc fit $Q=(\gamma+z)R$, et facto $z=-\alpha$ erit $Q=-\omega R$ si etiam in R ponatur $z=-\alpha$. Deinde cum sit

$$\frac{dQ}{dz}=R+\frac{(\gamma+z)dR}{dz}=R-\frac{\omega dR}{dz},$$

eodem casu erit

$$\frac{1}{dz}d.\frac{1}{Q}=-\frac{1}{\omega\omega R}+\frac{dR}{\omega RRdz}=-\frac{1}{\omega\omega}\cdot\frac{1}{R}-\frac{1}{\omega dz}d.\frac{1}{R}.$$

Quocirca ex factore quadrato $(\alpha+z)^2$ per praecedens problema haec obtinetur integralis pars:

$$-\frac{1}{\omega R}x^{-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx-\left(\frac{1}{\omega\omega R}+\frac{1}{\omega dz}d.\frac{1}{R}\right)x^{-\alpha}\int x^{\alpha}Xdx$$

cuius ambo membra in infinitum excrescunt ob $\omega=0$. Adiiciamus autem partem ex tertio factore

$$\gamma+z=\alpha-\omega+z$$

oriundam

oriundam, quae ob $\frac{P}{\gamma+z} = (\alpha+z)^{\epsilon} R$ eſt

$$\frac{1}{(\alpha+z)^{\epsilon} R} x^{-\gamma} \int x^{\gamma} X\, dx \text{ poſito } z = -\gamma = -\alpha + \omega.$$

Quod ſi iam R vt ante is fuerit valor, qui oritur poſito $z = -\alpha$; augendo hunc valorem particula ω, loco $\frac{1}{R}$ ſcribi debet

$$\frac{1}{R} + \frac{\omega}{dz} d. \frac{1}{R} + \frac{\omega^2}{1.2\, dz^2} dd. \frac{1}{R} \text{ etc.}$$

ſi quidem valorem $z = -\alpha$ et hic retineamus; vnde haec integralis pars ob $\alpha + z = \omega$ erit

$$\left(\frac{1}{\omega\omega R} + \frac{1}{\omega dz} d. \frac{1}{R} + \frac{1}{2 dz^2} dd. \frac{1}{R}\right) x^{-\alpha+\omega} \int x^{\alpha-\omega} X\, dx$$

ſicque manifeſtum eſt illum ipſius $\frac{1}{R}$ valorem vsque ad ſecundam poteſtatem ipſius ω continuari debuiſſe, atque eadem lege hic alteram partem x inuoluentem exprimi conueniet. Ad quod obſeruo ſi habeatur huiusmodi formula $x^{\omega} \int x^{-\omega} V\, dx$ ſecundum poteſtates ipſius ω euoluenda, id hac ratione commodiſſime fieri. Poſito

$$v = x^{\omega} \int x^{-\omega} V\, dx,$$

vt ſit

$$x^{-\omega} v = \int x^{-\omega} V\, dx$$

erit differentiando

$$dv - \frac{\omega v\, dx}{x} = V\, dx,$$

quare poſito

$$v = T + \omega T' + \omega^2 T'' + \omega^3 T''' \text{ etc}$$

 habe-

habebitur terminos ſecundum poteſtates ipſius ω diſponendo:

$$\left.\begin{array}{r} dT + \omega\, dT' + \omega\omega\, dT'' + \omega^3 dT''' + \text{etc.} \\ -V dx - \omega T \frac{dx}{x} - \omega\omega T' \frac{dx}{x} - \omega^3 T'' \frac{dx}{x} - \text{etc.} \end{array}\right\} = 0$$

ideoque

$$T = \int V dx;\quad T' = \int \tfrac{dx}{x} \int V dx;\quad T'' = \int \tfrac{dx}{x} \int \tfrac{dx}{x} \int V dx \text{ etc.}$$

Conſequenter cum in applicatione ſit $V = x^\alpha X$ erit pars integralis ex factore $\gamma + z = \alpha - \omega + z$ nata:

$$\left(\frac{1}{\omega\omega R} + \frac{1}{\omega dz} d.\frac{1}{R} + \frac{1}{2 dz^2} dd.\frac{1}{R}\right) x^{-\alpha} \left(\int x^\alpha X dx + \omega \int \tfrac{dx}{x} \int x^\alpha X dx \right.$$
$$\left. + \omega^2 \int \tfrac{dx}{x} \int \tfrac{dx}{x} \int x^\alpha X dx\right)$$

qua cum parte ex $(\alpha + z)^2$ nata iunctim ſumta omnia membra infinita ſe mutuo deſtruunt, et pro quantitatis $P = (\alpha + z)^3 R$ factore cubico $(\alpha + z)^3$ in integrale ingreditur haec pars:

$$\frac{1}{R} x^{-\alpha} \int \tfrac{dx}{x} \int \tfrac{dx}{x} \int x^\alpha X dx + \frac{1}{dz} d.\frac{1}{R} x^{-\alpha} \int \tfrac{dx}{x} \int x^\alpha X dx$$
$$+ \frac{1}{2 dz^2} dd.\frac{1}{R} x^{-\alpha} \int x^\alpha X dx$$

ſi modo in quantitate $R = \frac{P}{(\alpha + z)^3}$ vbique ſcribatur $z = -\alpha$.

Coroll. 1.

1272. Methodus in ſolutione huius problematis adhibita facile ad quotcunque factores aequales extendi poteſt. Si enim fuerit $(\alpha + z)^m$ factor quantitatis P, atque in hac fractione $\frac{(\alpha + z)^m}{P}$ ſuisque

que differentialibus, postquam fuerint euoluta ponatur $z = -\alpha$, partes integralis inde natae ita se habebunt

Factor quant. P	$\alpha + z$	$(\alpha + z)^2$	$(\alpha + z)^3$
Pars integralis	$\frac{\alpha+z}{P} x^{-\alpha} \int x^\alpha X dx$	$\frac{(\alpha+z)^2}{P} x^{-\alpha} \int \frac{dx}{x} \int x^\alpha X dx$	$\frac{(\alpha+z)^3}{P} x^{-\alpha} \int \frac{dx}{x} \int \frac{dx}{x} \int x^\alpha X dx$
		$\frac{1}{dz} d. \frac{(\alpha+z)^2}{P} x^{-\alpha} \int x^\alpha X dx$	$\frac{1}{dz} d. \frac{(\alpha+z)^3}{P} x^{-\alpha} \int \frac{dx}{x} \int x^\alpha X dx$
			$\frac{1}{2dz^2} dd. \frac{(\alpha+z)^3}{P} x^{-\alpha} \int x^\alpha X dx.$

Coroll. 2.

1273. Si fuerint duo pluresue factores duplices inter se aequales sumtis

$$\alpha = f(\text{cos.}\,\theta + \sqrt{-1}.\,\text{sin.}\,\theta) \text{ et } \beta = f(\text{cos.}\,\theta - \sqrt{-1}.\,\text{sin.}\,\theta),$$

partes pro $(\alpha+z)^n$ et $(\beta+z)^n$ seorsim euolutae methodo supra adhibita non difficulter coniungentur et ad realitatem reducentur.

Scholion.

1274. Simili methodo, qua hoc caput est pertractatum, in euolutione Capitis III. huius sectionis vti oportebat, neque tum vllum periculum in errores prolabendi fuisset pertimescendum. Superfluum autem nunc foret, errores ibi commissos hic emendare, cum non solum methodus plane esset eadem, sed etiam aequatio hic tractata facile in formam ibi consideratam transmutari queat et vicissim.

vbi ob rationes fupra demonftratas tres poftremi termini coalefcunt in $+\frac{\int x^{\delta} X dx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}$, ita vt fit integrale quaefitum

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha} X dx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}+\frac{x^{-\beta}\int x^{\beta} X dx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}$$
$$+\frac{x^{-\gamma}\int x^{\gamma} X dx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)}+\frac{x^{-\delta}\int x^{\delta} X dx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}$$

fiquidem omnes factores fint inter fe inaequales. Cafus autem quibus duo pluresue funt aequales in corollariis explorabimus.

Coroll. 1.

1253. Si fuerint duo factores aequales nempe $\delta=\gamma$ feu fi fit

$$P=N(\alpha+z)(\beta+z)(\gamma+z)^2$$

ex eadem forma pro X^{m} ante inuenta oritur integrale:

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha} X dx}{(\beta-\alpha)(\gamma-\alpha)^2}-\frac{x^{-\gamma}\int x^{\gamma} X dx}{(\beta-\alpha)(\gamma-\alpha)^2}$$
$$+\frac{x^{-\beta}\int x^{\beta} X dx}{(\alpha-\beta)(\gamma-\beta)^2}-\frac{x^{-\gamma}\int x^{\gamma} X dx}{(\alpha-\beta)(\gamma-\beta)^2}$$
$$+\frac{x^{-\gamma}\int \frac{dx}{x}\int x^{\gamma} X dx}{(\alpha-\gamma)(\beta-\gamma)}$$

vbi

vbi membra negatiua ita repraesentari possunt

$$\frac{x^{-\gamma}\int x^{\gamma}X\,dx}{(\alpha-\beta)}\left(\frac{1}{(\gamma-\alpha)^2}-\frac{1}{(\gamma-\beta)^2}\right).$$

Coroll. 2.

1254. Si fuerint tres factores aequales vt sit

$$P=N(\alpha+z)(\beta+z)^3$$

ideoque $\delta=\gamma=\beta$ ex formula §. 1248. inuenta colligitur integrale

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}X\,dx}{(\beta-\alpha)^3}-\frac{x^{-\beta}\int x^{\beta}X\,dx}{(\beta-\alpha)^3}-\frac{x^{-\beta}\int\frac{dx}{x}\int x^{\beta}X\,dx}{(\beta-\alpha)^2}$$

$$+\frac{x^{-\beta}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}X\,dx}{\alpha-\beta}$$

Coroll. 3.

1255. Si omnes quatuor factores fuerint aequales, vt sit

$P=N(\alpha+z)^4$ existente $\delta=\gamma=\beta=\alpha$

ex forma pro tribus aequalibus §. 1249. inuenta fit integrale

$$Nxy=x^{-\alpha}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\alpha}X\,dx.$$

Coroll. 4.

1256. Si habeatur $\beta=\alpha$ et $\delta=\gamma$, vt bini factores sint aequales scilicet $P=N(\alpha+z)^2(\gamma+z)^2$ ex §. 1247. vbi factores erant $(\alpha+z)^2(\gamma+z)$

 colli-

colligitur integrale:

$$Nxy=\frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\alpha}Xdx-\frac{x^{-\gamma}}{(\gamma-\alpha)^3}\int x^{\gamma-\alpha-1}dx\int x^{\alpha}Xdx$$

$$-\frac{x^{-\alpha}}{(\gamma-\alpha)^3}\int x^{\alpha}Xdx+\frac{x^{-\gamma}}{(\gamma-\alpha)^3}\int x^{\gamma}Xdx+\frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx$$

quae ob

$$\int x^{\gamma-\alpha-1}dx\int x^{\alpha}Xdx=\frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int x^{\alpha}Xdx-\frac{1}{\gamma-\alpha}\int x^{\gamma}Xdx$$

contrahitur in hanc formam:

$$Nxy=\frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\alpha}Xdx+\frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx$$

$$-\frac{2x^{-\alpha}}{(\gamma-\alpha)^3}\int x^{\alpha}Xdx-\frac{2x^{-\gamma}}{(\alpha-\gamma)^3}\int x^{\gamma}Xdx.$$

Problema 170.

1257. Proposita hac aequatione differentiali quinti gradus

$$X=Ay+\frac{Bxdy}{dx}+\frac{Cx^2ddy}{dx^2}+\frac{Dx^3d^3y}{dx^3}+\frac{Ex^4d^4y}{dx^4}+\frac{Nx^5d^5y}{dx^5}$$

eius integrale per formulas integrales simplices euolvere.

Solutio.

Cum hic sit quantitas algebraica formanda:

$$P=A+B(z-1)+C(z-1)(z-2)+\ldots+N(z-1)(z-2)(z-3)(z-4)(z-5)$$

statua-

ſtatuatur

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)(\varepsilon+z)$$

ac ſi hi factores omnes ſint inter ſe inaequales ex integrali praecedente nouam inſtituendo integrationem prodibit integrale quaeſitum:

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}X\,dx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha)} + \frac{x^{-\beta}\int x^{\beta}X\,dx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta)}$$
$$+ \frac{x^{-\gamma}\int x^{\gamma}X\,dx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)(\varepsilon-\gamma)} + \frac{x^{-\delta}\int x^{\delta}X\,dx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)(\varepsilon-\delta)}$$
$$+ \frac{x^{-\varepsilon}\int x^{\varepsilon}X\,dx}{(\alpha-\varepsilon)(\beta-\varepsilon)(\gamma-\varepsilon)(\delta-\varepsilon)}$$

caſus quo duo pluresue factores ſunt aequales in Coroll. euoluemus.

Coroll. 1.

1258. Si fuerint duo factores aequales vt ſit $P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)^2$ ideoque $\varepsilon = \delta$ ex praecedente problemate colligitur integrale

$$Nxy = +\frac{x^{-\alpha}\int x^{\alpha}X\,dx - x^{-\delta}\int x^{\delta}X\,dx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)^2}$$
$$+ \frac{x^{-\beta}\int x^{\beta}X\,dx - x^{-\delta}\int x^{\delta}X\,dx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)^2}$$
$$+ \frac{x^{-\gamma}\int x^{\gamma}X\,dx - x^{-\delta}\int x^{\delta}X\,dx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)^2}$$
$$+ \frac{x^{-\delta}\int\frac{dx}{x}\int x^{\delta}X\,dx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}.$$

Coroll. 2.

1259. Si fuerint tres factores aequales, vt sit $P = N(\alpha+z)(\beta+z)(\gamma+z)^3$ ideoque $\varepsilon = \delta = \gamma$ ex Coroll. 1. probl. praeced. colligitur:

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}Xdx - x^{-\gamma}\int x^{\gamma}Xdx}{(\beta-\alpha)(\gamma-\alpha)^3} - \frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}Xdx}{(\beta-\alpha)(\gamma-\alpha)^2}$$

$$+ \frac{x^{-\beta}\int x^{\beta}Xdx - x^{-\gamma}\int x^{\gamma}Xdx}{(\alpha-\beta)(\gamma-\beta)^3} - \frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}Xdx}{(\alpha-\beta)(\gamma-\beta)^2}$$

$$+ \frac{x^{-\gamma}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}Xdx}{(\alpha-\gamma)(\beta-\gamma)}.$$

Coroll. 3.

1260. Si quatuor factores sint aequales, vt sit $P = N(\alpha+z)(\beta+z)^4$ ideoque $\varepsilon = \delta = \gamma = \beta$ erit per (1254)

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}Xdx}{(\beta-\alpha)^4} - \frac{x^{-\beta}\int x^{\beta}Xdx}{(\beta-\alpha)^4} - \frac{x^{-\beta}\int\frac{dx}{x}\int x^{\beta}Xdx}{(\beta-\alpha)^3}$$

$$- \frac{x^{-\beta}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx}{(\beta-\alpha)^2} + \frac{x^{-\beta}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx}{\alpha-\beta}$$

ac si omnes quinque sint inter se aequales seu

$$P = N(\alpha+z)^5$$

erit integrale

$$Nxy = x^{-\alpha}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\alpha}Xdx.$$

Coroll. 4.

Coroll. 4.

1261. Si P habeat duos factores quadratos, vt fit

$P = N(\alpha+z)(\beta+z)^2(\gamma+z)^2$ ideoque $\delta=\gamma$ et $\varepsilon=\beta$

erit ex §. 1253. integrale reductione necessaria facta

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}X\,dx - x^{-\beta}\int x^{\beta}X\,dx}{(\beta-\alpha)^2(\gamma-\alpha)^2} - \frac{x^{-\gamma}\int x^{\gamma}X\,dx + x^{-\beta}\int x^{\beta}X\,dx}{(\beta-\alpha)(\alpha-\gamma)^2(\beta-\gamma)}$$

$$+ \frac{x^{-\beta}\int\frac{dx}{x}\int x^{\beta}X\,dx}{(\alpha-\beta)(\gamma-\beta)^2} - \frac{x^{-\gamma}\int x^{\gamma}X\,dx + x^{-\beta}\int x^{\beta}X\,dx}{(\alpha-\beta)(\beta-\gamma)^3}$$

$$+ \frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}X\,dx}{(\alpha-\gamma)(\beta-\gamma)^2} - \frac{x^{-\gamma}\int x^{\gamma}X\,dx + x^{-\beta}\int x^{\beta}X\,dx}{(\alpha-\gamma)(\beta-\gamma)^3}$$

quae porro redigitur ad hanc formam

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}X\,dx}{(\beta-\alpha)^2(\gamma-\alpha)^2} + \frac{x^{-\beta}\int\frac{dx}{x}\int x^{\beta}X\,dx}{(\alpha-\beta)(\gamma-\beta)^2} - \frac{x^{-\beta}\int x^{\beta}X\,dx}{(\alpha-\beta)^2(\gamma-\beta)^2} - \frac{2x^{-\beta}\int x^{\beta}X\,dx}{(\alpha-\beta)(\gamma-\beta)^3}$$

$$+ \frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}X\,dx}{(\alpha-\gamma)(\beta-\gamma)^2} - \frac{x^{-\gamma}\int x^{\gamma}X\,dx}{(\alpha-\gamma)^2(\beta-\gamma)^2} - \frac{2x^{-\gamma}\int x^{\gamma}X\,dx}{(\alpha-\gamma)(\beta-\gamma)^3}.$$

Coroll. 5.

1262. Si P habeat et factorem quadratum et cubicum vt fit

$P = N(\alpha+z)^2(\gamma+z)^3$, ideoque $\beta=\alpha$ et $\varepsilon=\delta=\gamma$

ex §. 1254. colligitur integrale:

$$Nxy = \frac{x^{-\alpha}\int\frac{dx}{x}\int x^{\alpha}X\,dx}{(\gamma-\alpha)^3} - \frac{3x^{-\alpha}\int x^{\alpha}X\,dx}{(\gamma-\alpha)^4}$$

$$+ \frac{x^{-\gamma}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}X\,dx}{(\alpha-\gamma)^2} - \frac{2x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}X\,dx}{(\alpha-\gamma)^3} - \frac{3x^{-\gamma}\int x^{\gamma}X\,dx}{(\alpha-\gamma)^4}.$$

Scholion.

Scholion.

1263. Ex his formulis parum constat, quemadmodum eas vlterius pro maiori factorum numero continuari oporteat, si quidem factorum aliquot inter se fuerint aequales; integralium enim partes, quae factoribus inaequalibus respondent, legem seruant manifestam. Quae autem partibus aequalibus respondent, adhibita certa reductione commodius exprimi possunt. Veluti pro casu Coroll. 1. si breuitatis gratia ponatur $\alpha-\delta=p$, $\beta-\delta=q$ et $\gamma-\delta=r$ forma $x^{-\delta}\int x^{\delta}X\,dx$ ducta est in:

$$\frac{1}{(p-q)(r-p)pp}+\frac{1}{(p-q)(q-r)qq}+\frac{1}{(r-p)(q-r)rr}$$

seu $\frac{(q-r)qqrr+(r-p)pprr+(p-q)ppqq}{(p-q)(q-r)(r-p)ppqqrr}$

cuius fractionis numerator est

$$-(p-q)(q-r)(r-p)(pq+pr+qr)$$

ita vt haec fractio reducatur ad istam

$$\frac{-pq-pr-qr}{ppqqrr}=-\frac{1}{pqr}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right).$$

Quando ergo est

$$P=N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)^{2}$$

integrale ita se habet:

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}X\,dx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)^{2}}+\frac{x^{-\beta}\int x^{\beta}X\,dx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)^{2}}+\frac{x^{-\gamma}\int x^{\gamma}X\,dx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)^{2}}$$
$$+\frac{x^{-\delta}\int\frac{dx}{x}\int x^{\delta}X\,dx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}-\frac{x^{-\delta}\int x^{\delta}X\,dx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\left(\frac{1}{\alpha-\delta}+\frac{1}{\beta-\delta}+\frac{1}{\gamma-\delta}\right).$$

Pro

Pro casu autem $P=N(\alpha+z)(\beta+z)(\gamma+z)^2$ habebitur

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}Xdx}{(\beta-\alpha)(\gamma-\alpha)^2}+\frac{x^{-\beta}\int x^{\beta}Xdx}{(\alpha-\beta)(\gamma-\beta)^2}$$

$$+\frac{x^{-\gamma}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}Xdx}{(\alpha-\gamma)(\beta-\gamma)}-\frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}Xdx}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{\alpha-\gamma}+\frac{1}{\beta-\gamma}\right)$$

$$+\frac{x^{-\gamma}\int x^{\gamma}Xdx}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{(\alpha-\gamma)^2}+\frac{1}{(\alpha-\gamma)(\beta-\gamma)}+\frac{1}{(\beta-\gamma)^2}\right).$$

Tum vero pro casu $P=N(\alpha+z)(\beta+z)^4$ fit

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}Xdx}{(\beta-\alpha)^4}+\frac{x^{-\beta}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx}{\alpha-\beta}$$

$$-\frac{x^{-\beta}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx}{\alpha-\beta}\cdot\frac{1}{\alpha-\beta}+\frac{x^{-\beta}\int\frac{dx}{x}\int x^{\beta}Xdx}{\alpha-\beta}\cdot\frac{1}{(\alpha-\beta)^2}-\frac{x^{-\beta}\int x^{\beta}Xdx}{\alpha-\beta}\cdot\frac{1}{(\alpha-\beta)^3}.$$

At pro casu $P=(\alpha+z)(\beta+z)^2(\gamma+z)^2$ integrale ita se habet

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}Xdx}{(\beta-\alpha)^2(\gamma-\alpha)^2}+\frac{x^{-\beta}\int\frac{dx}{x}\int x^{\beta}Xdx}{(\alpha-\beta)(\gamma-\beta)^2}-\frac{x^{-\beta}\int x^{\beta}Xdx}{(\alpha-\beta)(\gamma-\beta)^2}\left(\frac{1}{\alpha-\beta}+\frac{2}{\gamma-\beta}\right)$$

$$+\frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}Xdx}{(\alpha-\gamma)(\beta-\gamma)^2}-\frac{x^{-\gamma}\int x^{\gamma}Xdx}{(\alpha-\gamma)(\beta-\gamma)^2}\left(\frac{1}{\alpha-\gamma}+\frac{2}{\beta-\gamma}\right).$$

Denique pro casu $P=N(\alpha+z)^2(\gamma+z)^3$ est

$$Nxy=\frac{x^{-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx}{(\gamma-\alpha)^3}-\frac{x^{-\alpha}\int x^{\alpha}Xdx}{(\gamma-\alpha)^3}\cdot\frac{3}{\gamma-\alpha}$$

$$+\frac{x^{-\gamma}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}Xdx}{(\alpha-\gamma)^2}-\frac{x^{-\gamma}\int\frac{dx}{x}\int x^{\gamma}Xdx}{(\alpha-\gamma)^2}\cdot\frac{2}{\alpha-\gamma}+\frac{x^{-\gamma}\int x^{\gamma}Xdx}{(\alpha-\gamma)^2}\cdot\frac{3}{(\alpha-\gamma)^2}.$$

vnde indoles harum formularum iam magis fit perspicua, simulque patet partem integralis ex aliquot factoribus oriundam non pendere ab aequalitate reliquorum. Quocirca iam problema generale aggredi licebit.

Problema 171.

1264. Proposita aequatione differentiali cuiuscunque gradus huius formae:

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \dots + \frac{Nx^nd^ny}{dx^n}$$

ex qua forma algebraica hac lege formata

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \dots$$
$$\dots + N(z-1)(z-2)\dots\dots(z-n)$$

omnes factores habeat inter se inaequales; valorem ipsius y completum per formulas integrales simplices exhibere.

Solutio.

Sint primo formae P omnes factores simplices reales:

$$P = N(\alpha+z)(\beta+z)(\gamma+z) \dots (\nu+z)$$

factorum numero existente $= n$, et ex antecedentibus patet ex quolibet factore nasci integralis partem. Ad has partes inueniendas, eliciantur sequentes valores:

1) posito $z = -\alpha$ fit $\mathfrak{A} = \frac{P}{\alpha + z}$ seu $\mathfrak{A} = \frac{dP}{dz}$

2) posito $z = -\beta$ fit $\mathfrak{B} = \frac{P}{\beta + z}$ seu $\mathfrak{B} = \frac{dP}{dz}$

3) posito $z = -\gamma$ fit $\mathfrak{C} = \frac{P}{\gamma + z}$ seu $\mathfrak{C} = \frac{dP}{dz}$

etc.

Cum

Cum igitur ſit

$$(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\ldots(\nu-\alpha)=\frac{\mathfrak{A}}{N},$$

littera N ex ſuperioribus formis per diuiſionem tolletur, fietque integrale quaeſitum:

$$xy=\tfrac{1}{\mathfrak{A}}x^{-\alpha}\int x^{\alpha}X\,dx+\tfrac{1}{\mathfrak{B}}x^{-\beta}\int x^{\beta}X\,dx+\tfrac{1}{\mathfrak{C}}x^{-\gamma}\int x^{\gamma}X\,dx+\text{etc.}$$

quoad ſinguli factores fuerint exhauſti.

Quodſi iam forma P factores habeat imaginarios, partium inde ortarum imaginariarum ad realitatem reductio ſequenti modo inſtituetur. Quoniam bini factores ſimplices imaginarii praebent factorem duplicem realem, ponamus

$$(\alpha+z)(\beta+z)=ff+2fz\,\text{cof.}\,\theta+zz$$

ita vt ſit

$$\alpha=f(\text{cof.}\,\theta+\sqrt{-1}.\text{ſin.}\,\theta) \text{ et } \beta=f(\text{cof.}\,\theta-\sqrt{-1}.\text{ſin.}\,\theta)$$

vnde primum valores literarum $\mathfrak{A}$ et $\mathfrak{B}$ definiantur, quarum cum vtraque deriuetur ex forma $\frac{dP}{dz}$, illa poſito $z=-\alpha$ haec vero poſito $z=-\beta$; in ipſa forma $\frac{dP}{dz}$ loco z vbique ſcribatur

$$-f(\text{cof.}\,\theta\pm\sqrt{-1}.\,\text{ſin.}\,\theta),$$

prodeatque $\mathfrak{P}\pm\mathfrak{Q}\sqrt{-1}$; ac perſpicuum eſt fore

$$\mathfrak{A}=\mathfrak{P}+\mathfrak{Q}\sqrt{-1} \quad\text{et}\quad \mathfrak{B}=\mathfrak{P}-\mathfrak{Q}\sqrt{-1},$$

vbi notandum eſt, quantitates $\mathfrak{P}$ et $\mathfrak{Q}$ eſſe reales. Deinde cum ſit

$$x^{m+n\sqrt{-1}}=x^{m}e^{n\sqrt{-1}.lx}=x^{m}(\text{cof.}(nlx)+\sqrt{-1}.\text{ſin.}(nlx))$$

si breuitatis ergo ponamus angulum $f\,\mathrm{sin.}\,\theta.\,lx = \Phi$ erit:

$$x^{\alpha} = x^{f\cos.\theta}(\mathrm{cos.}\,\Phi + \sqrt{-1}.\,\mathrm{sin.}\,\Phi);\quad x^{-\alpha} = x^{-f\cos.\theta}(\mathrm{cos.}\,\Phi - \sqrt{-1}.\,\mathrm{si.}\,\Phi)$$

$$x^{\beta} = x^{f\cos.\theta}(\mathrm{cos.}\,\Phi - \sqrt{-1}.\,\mathrm{sin.}\,\Phi);\quad x^{-\beta} = x^{-f\cos.\theta}(\mathrm{cs.}\,\Phi + \sqrt{-1}.\,\mathrm{si.}\,\Phi).$$

Quare pro binis partibus

$$\frac{1}{\mathfrak{A}} x^{-\alpha}\int x^{\alpha} X\, dx + \frac{1}{\mathfrak{B}} x^{-\beta}\int x^{\beta} X\, dx$$

ob $\mathfrak{A}\mathfrak{B} = \mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}$ habebimus:

$$+\frac{x^{-f\cos.\theta}}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}\left\{\begin{matrix}(\mathfrak{P}-\mathfrak{Q}\sqrt{-1})(\mathrm{cs.}\,\Phi - \sqrt{-1}.\,\mathrm{si.}\,\Phi)\int x^{f\cos.\theta}(\mathrm{cs.}\,\Phi + \sqrt{-1}.\,\mathrm{si.}\,\Phi) X dx \\ (\mathfrak{P}+\mathfrak{Q}\sqrt{-1})(\mathrm{cs.}\,\Phi + \sqrt{-1}.\,\mathrm{si.}\,\Phi)\int x^{f\cos.\theta}(\mathrm{cs.}\,\Phi - \sqrt{-1}.\,\mathrm{si.}\,\Phi) X dx\end{matrix}\right\}$$

quae forma ob partes imaginarias se tollentes reducitur ad hanc:

$$\frac{2x^{-f\cos.\theta}(\mathfrak{P}\,\mathrm{cs.}\,\Phi - \mathfrak{Q}\,\mathrm{si.}\,\Phi)\int x^{f\cos.\theta} X dx\,\mathrm{cs.}\,\Phi + 2x^{-f\cos.\theta}(\mathfrak{Q}\,\mathrm{cs.}\,\Phi + \mathfrak{P}\,\mathrm{si.}\,\Phi)\int x^{f\cos.\theta} X dx\,\mathrm{si.}\,\Phi}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}.$$

Talisque forma ad integrale accedit, quoties forma P huiusmodi habet factorem duplicem $ff + 2fz\cos.\theta + zz$.

Coroll. 1.

1265. Etsi autem factorum simplicium ipsius P quidam sunt imaginarii, eorum qui sunt reales euolutio inde non perturbatur, sed ex singulis partes in integrale inferendae a natura reliquorum factorum minime pendent.

Coroll. 2.

1266. Pars integralis ex binis factoribus imaginariis seu vno factore duplici oriunda aliquanto succinctius

cinctius repraesentari potest si ponatur

$$\mathfrak{P} = \mathfrak{D} \cos. \zeta \quad \text{et} \quad \mathfrak{Q} = \mathfrak{D} \sin. \zeta,$$

sic enim ea fiet:

$$\tfrac{a}{\mathfrak{D}} x^{-f\cos.\theta} (\cos.(\zeta+\Phi) \int x^{f\cos.\theta} X dx \cos.\Phi + \sin.(\zeta+\Phi) \int x^{f\cos.\theta} X dx \sin.\Phi)$$

vbi ζ et θ sunt anguli constantes Φ vero variabilis ob $\Phi = f \sin.\theta . lx$.

Problema 172.

1267. Si pro aequatione differentiali in praec. probl. proposita quantitas algebraica P inde formata duos habeat factores simplices aequales, integralis partem inde oriundam inuestigare.

Solutio.

In forma ergo ante exhibita:

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z) \text{ etc.}$$

ponamus esse $\beta = \alpha$, quoniam vero tum vtraque integralis pars oritur infinita, altera signo +, altera signo − affecta, ita vt iunctim sumtae partem constituant finitam, ad hanc eliciendam statuamus $\beta = \alpha - \omega$ denotante ω quantitatem euanescentem, eritque

$$\mathfrak{A} = -N\omega(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha) \text{ etc. et}$$

$$\mathfrak{B} = +N\omega(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta) \text{ etc.}$$

Ponatur iam

$$\frac{P}{(\alpha+z)(\beta+z)} = \frac{P}{(\alpha+z)^2} = Q$$

 vt

vt ſit

$$Q = N(\gamma + z)(\delta + z)(\varepsilon + z) \text{ etc.}$$

ac manifeſtum eſt fieri

$$\mathfrak{A} = -\omega Q \text{ poſito } z = -\alpha \text{ et}$$

$$\mathfrak{B} = \omega Q \text{ poſito } z = -\beta = -\alpha + \omega$$

vnde intelligitur valorem ipſius Q poſteriorem excedere priorem ſuo differentiali dQ ſi fiat

$$z = -\alpha \text{ et } dz = \omega,$$

ita vt ſit

$$\mathfrak{B} = \omega(Q + \omega.\tfrac{dQ}{dz}) \text{ poſito } z = -\alpha,$$

hincque

$$\tfrac{1}{\mathfrak{B}} = \tfrac{1}{\omega Q} - \tfrac{dQ}{QQdz} = \tfrac{1}{\omega Q} + \tfrac{1}{dz}d.\tfrac{1}{Q}, \text{ exiſtente } \tfrac{1}{\mathfrak{A}} = -\tfrac{1}{\omega Q}$$

tum vero cum ſit

$$x^{\beta} = x^{\alpha}x^{-\omega} = x^{\alpha}(1 - \omega lx) \text{ et } x^{-\beta} = x^{-\alpha}(1 + \omega lx),$$

binae partes integralis quaeſitae erunt

$$-\tfrac{1}{\omega Q}x^{-\alpha}\int x^{\alpha}Xdx + (\tfrac{1}{\omega Q} + \tfrac{1}{dz}d.\tfrac{1}{Q})x^{-\alpha}(1 + \omega lx)\int x^{\alpha}Xdx(1 - \omega lx)$$

vbi cum membra per ω diuiſa ſe deſtruant, reſultat

$$\tfrac{1}{Q}x^{-\alpha}(lx\int x^{\alpha}Xdx - \int x^{\alpha}Xdxlx) + \tfrac{1}{dz}d.\tfrac{1}{Q}.x^{-\alpha}\int x^{\alpha}Xdx$$

ſeu $\tfrac{1}{Q}x^{-\alpha}\int \tfrac{dx}{x}\int x^{\alpha}Xdx + \tfrac{1}{dz}d.\tfrac{1}{Q}.x^{-\alpha}\int x^{\alpha}Xdx$

ſiquidem tam in valore $\tfrac{1}{Q}$ quam in $\tfrac{1}{dz}d.\tfrac{1}{Q}$ vbique loco z ſcribatur $-\alpha$. Cum vero ſit $Q = \tfrac{P}{(\alpha + z)^2}$, hi valores inde facile inueniuntur.

Coroll. 1.

Coroll. 1.

1268. Quodſi ergo quantitas algebraica P ex aequatione differentiali formata factorem habeat quadratum $(\alpha + z)^2$, inde in integrale transferenda eſt haec portio

$$\frac{(\alpha + z)^2}{P} x^{-\alpha} \int \frac{dx}{x} \int x^\alpha X dx + \frac{1}{dz} d. \frac{(\alpha + z)^2}{P} x^{-\alpha} \int x^\alpha X dx$$

poſito $z = -\alpha$; dum ſi hic factor $\alpha + z$ eſſet ſolitarius integralis pars inde oriunda foret:

$$\frac{\alpha + z}{P} x^{-\alpha} \int x^\alpha X dx \text{ poſito } z = -\alpha.$$

Coroll. 2.

1269. Cum ſit $Q = \frac{P}{(\alpha + z)^2}$, caſu $z = -\alpha$ fiet $Q = \frac{ddP}{2dz^2}$ verum quia hic ipſi z iam valor determinatus eſt tributus hinc $\frac{dQ}{dz}$ colligere non licet, ſed prima eſt vtendum qua fit $\frac{dQ}{dz} = \frac{(\alpha + z)dP - 2Pdz}{(\alpha + z)^3 dz}$, cuius fractionis cum numerator et denominator caſu $z = -\alpha$ euaneſcat, erit pro eodem caſu

$$\frac{dQ}{dz} = \frac{(\alpha + z)ddP - dzdP}{3(\alpha + z)^2 dz^2} = \frac{(\alpha + z)d^3P}{6(\alpha + z)dz^3} = \frac{d^3P}{6dz^3}.$$

Coroll. 3.

1270. Hoc valore inuento, quia eſt eodem caſu $z = -\alpha$ quantitas $Q = \frac{ddP}{2dz^2}$ erit

$$\frac{1}{dz} d. \frac{1}{Q} = -\frac{dQ}{QQdz} = -\frac{2dzd^3P}{3ddP^2} \text{ ſeu}$$

$$\frac{1}{dz} d. \frac{1}{Q} = \frac{2dz}{3} d. \frac{1}{ddP},$$

ex quibus formulis, si factores ipsius P non sint euoluti partes integralis facilius reperiuntur.

Problema 173.

1271. Si pro aequatione differentiali praecedente quantitas algebraica P inde formata factorem habeat cubicum $(\alpha+z)^3$ integralis partem inde oriundam inuestigare.

Solutio.

Ponamus ergo esse

$$P=(\alpha+z)^2(\gamma+z)R \text{ existente } \gamma=\alpha-\omega,$$

vbi ω pro quantitate euanescente assumitur. Quod ergo ante erat Q id hinc fit $Q=(\gamma+z)R$, et facto $z=-\alpha$ erit $Q=-\omega R$ si etiam in R ponatur $z=-\alpha$. Deinde cum sit

$$\frac{dQ}{dz}=R+\frac{(\gamma+z)dR}{dz}=R-\frac{\omega dR}{dz},$$

eodem casu erit

$$\frac{1}{dz}d.\frac{1}{Q}=-\frac{1}{\omega\omega R}+\frac{dR}{\omega RRdz}=-\frac{1}{\omega\omega}.\frac{1}{R}-\frac{1}{\omega dz}d.\frac{1}{R}.$$

Quocirca ex factore quadrato $(\alpha+z)^2$ per praecedens problema haec obtinetur integralis pars:

$$-\frac{1}{\omega R}x^{-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx-\left(\frac{1}{\omega\omega R}+\frac{1}{\omega dz}d.\frac{1}{R}\right)x^{-\alpha}\int x^{\alpha}Xdx$$

cuius ambo membra in infinitum excrescunt ob $\omega=0$. Adiiciamus autem partem ex tertio factore

$$\gamma+z=\alpha-\omega+z$$

oriundam

oriundam, quae ob $\frac{P}{\gamma+z} = (\alpha + z)^{\epsilon} R$ eſt

$$\frac{1}{(\alpha+z)^2 R} x^{-\gamma} \int x^{\gamma} X \, dx \text{ poſito } z = -\gamma = -\alpha + \omega.$$

Quod ſi iam R vt ante is fuerit valor, qui oritur poſito $z = -\alpha$; augendo hunc valorem particula ω, loco $\frac{1}{R}$ ſcribi debet

$$\frac{1}{R} + \frac{\omega}{dz} d. \frac{1}{R} + \frac{\omega^2}{1.2\, dz^2} dd. \frac{1}{R} \text{ etc.}$$

ſi quidem valorem $z = -\alpha$ et hic retineamus: vnde haec integralis pars ob $\alpha + z = \omega$ erit

$$\left(\frac{1}{\omega\omega R} + \frac{1}{\omega\, dz} d. \frac{1}{R} + \frac{1}{2\, dz^2} dd. \frac{1}{R}\right) x^{-\alpha+\omega} \int x^{\alpha-\omega} X \, dx$$

ſicque manifeſtum eſt illum ipſius $\frac{1}{R}$ valorem vsque ad ſecundam poteſtatem ipſius ω continuari debuiſſe, atque eadem lege hic alteram partem x inuoluentem exprimi conueniet. Ad quod obſeruo ſi habeatur huiusmodi formula $x^{\omega} \int x^{-\omega} V \, dx$ ſecundum poteſtates ipſius ω euoluenda, id hac ratione commodiſſime fieri. Poſito

$$v = x^{\omega} \int x^{-\omega} V \, dx,$$

vt ſit

$$x^{-\omega} v = \int x^{-\omega} V \, dx$$

erit differentiando

$$dv - \frac{\omega v \, dx}{x} = V \, dx,$$

quare poſito

$$v = T + \omega T' + \omega^2 T'' + \omega^3 T''' \text{ etc}$$

 habe-

habebitur terminos ſecundum poteſtates ipſius ω diſponendo :

$$\left.\begin{matrix} dT + \omega\, dT' + \omega\omega\, dT'' + \omega^3 dT''' + \text{etc.} \\ -V dx - \omega T\frac{dx}{x} - \omega\omega T'\frac{dx}{x} - \omega^3 T''\frac{dx}{x} - \text{etc.} \end{matrix}\right\} = 0$$

ideoque

$$T = \int V dx\,;\ T' = \int\frac{dx}{x}\int V dx\,;\ T'' = \int\frac{dx}{x}\int\frac{dx}{x}\int V dx \text{ etc.}$$

Conſequenter cum in applicatione ſit $V = x^\alpha X$ erit pars integralis ex factore $\gamma + z = \alpha - \omega + z$ nata :

$$\left(\frac{1}{\omega\omega R} + \frac{1}{\omega dz} d.\frac{1}{R} + \frac{1}{2\, dz^2} dd.\frac{1}{R}\right) x^{-\alpha} \left(\int x^\alpha X dx + \omega\int\frac{dx}{x}\int x^\alpha X dx \right.$$
$$\left. + \omega^2\int\frac{dx}{x}\int\frac{dx}{x}\int x^\alpha X dx\right)$$

qua cum parte ex $(\alpha + z)^3$ nata iunctim ſumta omnia membra infinita ſe mutuo deſtruunt, et pro quantitatis $P = (\alpha + z)^3 R$ factore cubico $(\alpha + z)^3$ in integrale ingreditur haec pars:

$$\frac{1}{R} x^{-\alpha}\int\frac{dx}{x}\int\frac{dx}{x}\int x^\alpha X dx + \frac{1}{dz} d.\frac{1}{R} x^{-\alpha}\int\frac{dx}{x}\int x^\alpha X dx$$
$$+ \frac{1}{2\, dz^2} dd.\frac{1}{R} x^{-\alpha}\int x^\alpha X dx$$

ſi modo in quantitate $R = \frac{P}{(\alpha + z)^3}$ vbique ſcribatur $z = -\alpha$.

Coroll. I.

1272. Methodus in ſolutione huius problematis adhibita facile ad quotcunque factores aequales extendi poteſt. Si enim fuerit $(\alpha + z)^m$ factor quantitatis P, atque in hac fractione $\frac{(\alpha + z)^m}{P}$ fuis-

que

que differentialibus, postquam fuerint euoluta ponatur $z = -\alpha$, partes integralis inde natae ita se habebunt

Factor quant. P	$\alpha + z$	$(\alpha + z)^2$	$(\alpha + z)^3$
Pars integralis	$\frac{\alpha + z}{P} x^{-\alpha} \int x^{\alpha} X dx$	$\frac{(\alpha+z)^2}{P} x^{-\alpha} \int \frac{dx}{x} \int x^{\alpha} X dx$	$\frac{(\alpha+z)^3}{P} x^{-\alpha} \int \frac{dx}{x} \int \frac{dx}{x} \int x^{\alpha} X dx$
		$\frac{1}{dz} d. \frac{(\alpha+z)^2}{P} x^{-\alpha} \int x^{\alpha} X dx$	$\frac{1}{dz} d. \frac{(\alpha+z)^3}{P} x^{-\alpha} \int \frac{dx}{x} \int x^{\alpha} X dx$
			$\frac{1}{2dz^2} dd. \frac{(\alpha+z)^3}{P} x^{-\alpha} \int x^{\alpha} X dx.$

Coroll. 2.

1273. Si fuerint duo pluresue factores duplices inter se aequales sumtis

$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta)$ et $\beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta)$, partes pro $(\alpha + z)^2$ et $(\beta + z)^2$ seorsim euolutae methodo supra adhibita non difficulter coniungentur et ad realitatem reducentur.

Scholion.

1274. Simili methodo, qua hoc caput est pertractatum, in euolutione Capitis III. huius sectionis vti oportebat, neque tum vllum periculum in errores prolabendi fuisset pertimescendum. Superfluum autem nunc foret, errores ibi commissos hic emendare, cum non solum methodus plane esset eadem, sed etiam aequatio hic tractata facile in formam ibi consideratam transmutari queat et vicissim.

Quodſi enim in aequatione Cap. III.

$$X = Ay + \frac{B\,dy}{dx} + \frac{C\,ddy}{dx^2} + \frac{D\,d^3y}{dx^3} + \frac{E\,d^4y}{dx^4} + \text{etc.}$$

ſtatuatur $x = lv$, vt ſit $dx = \frac{dv}{v}$, functio autem X abeat in functionem ipſius v quae ſit V, proueniet aequatio eius formae quam hic tractauimus. Dum autem ibi elementum dx pro conſtanti eſt habitum, ad hanc conditionem exuendam ponamus

$$dy = p\,dx,\ dp = q\,dx,\ dq = r\,dx,\ dr = s\,dx \text{ etc.}$$

vt haec aequatio reſultet:

$$X = V = Ay + Bp + Cq + Dr + Es + \text{etc.}$$

Nunc autem ob $dx = \frac{dv}{v}$, adipiſcimur elemento dv conſtante ſumto:

$$p = \frac{dy}{dx} = \frac{v\,dy}{dv}$$

$$q = \frac{dp}{dx} = \frac{vv\,ddy}{dv^2} + \frac{v\,dy}{dv}$$

$$r = \frac{dq}{dx} = \frac{v^3\,d^3y}{dv^3} + \frac{3v^2\,ddy}{dv^2} + \frac{v\,dy}{dv}$$

$$s = \frac{dr}{dx} = \frac{v^4\,d^4y}{dv^4} + \frac{6v^3\,d^3y}{dv^3} + \frac{7vv\,ddy}{dv^2} + \frac{v\,dy}{dv}$$

$$t = \frac{ds}{dx} = \frac{v^5\,d^5y}{dv^5} + \frac{10v^4\,d^4y}{dv^4} + \frac{25v^3\,d^3y}{dv^3} + \frac{15vv\,ddy}{dv^2} + \frac{v\,dy}{dv}.$$

Quare

Quare aequatio inter v et y erit haec

$$\begin{array}{llllll} V = Ay + \frac{Bvdy}{dv} & + \frac{Cvvddy}{dv^2} & + \frac{Dv^3d^3y}{dv^3} & + \frac{Ev^4d^4y}{dv^4} & + \frac{Fv^5d^5y}{dv^5} \\ & + C & + 3D & + 6E & + 10F \\ & & + D & + 7E & + 25F \\ & & & + E & + 15F \\ & & & & + F \end{array}$$

cuius integrationem hic docuimus. Imprimis autem notandum est quantitatem algebraicam P hinc formandam

$$\begin{aligned} P = A &+ (B + C + D + E + F)(z-1) \\ &+ (C + 3D + 7E + 15F)(z-1)(z-2) \\ &+ (D + 6E + 25F)(z-1)(z-2)(z-3) + \text{etc.} \end{aligned}$$

ad hanc formam reduci:

$$\begin{aligned} P = A + B(z-1) + C(z-1)^2 + D(z-1)^3 + E(z-1)^4 \\ + F(z-1)^5 + \text{etc.} \end{aligned}$$

quae quantitas algebraica ab illa, qua in Cap. III. ad integrationem sumus vsi, hoc tantum differt, quod ibi littera z id quod hic formula $z-1$ expressimus; ex quo etiam ambarum integratio facillime altera ad alteram reducitur.

Conclusio Libri Primi.

1275. Atque haec fere sunt, quae ad librum primum de Calculo Integrali pertinere sunt visa, vbi methodum tradere institui functiones vnius variabilis ex data quacunque differentialium cuiusque ordinis relatione inuestigandi quod opus mihi equidem ita pertractasse videor, vt vix quicquam eorum, quae adhuc de hoc argumento ab aliis sunt inuenta et in medium allata, sit praetermissum.

ERRATA

ERRATA TYPOGRAPHICA

VOL. II[di.]

Pag. 6. lin. 9. loco quae omnia ad dx reducta lege quibus omnibus ad dx reductis.

pag. 12. lin. 7. loco $p = \int adp$ lege $y = \int adp$

pag. 14. lin. 14. loco colligemus lege colligimus.

pag. 17. lin. 9. loco forma legatur et forma.

pag. 21. lin. 12. loco praecedenti lege procedenti.

pag. 29. lin. 15. loco eandem lege eundem.

pag. 38. lin. 3. loco $= \frac{pdpdx}{1 - pp}$ lege $- \frac{pdpdx}{1 + pp}$.

— — — lin. 8. — diffentiata lege differentiata.

pag. 39. lin. 2. loco $\frac{bdq}{\sqrt{(1+aaqq)^{\frac{3}{2}}}}$ lege $\frac{bdq}{(1+aaqq)^{\frac{3}{2}}}$.

pag. 46. lin. 2. loco $\alpha - b = \frac{1}{y}(\beta e^{-\beta x} - a e^{-\alpha x})$

legatur — — — — $\alpha - \beta = \frac{1}{y}(b e^{-\beta x} - a e^{-\alpha x})$.

pag. 50. lin. 10. loco $\frac{du(1 + \sqrt{(1+uu)})}{u(uu+1)}$ lege $\frac{-du(1 + \sqrt{(1+uu)})}{u(uu+1)}$

et lin. 11. loco $\frac{du(1 + \sqrt{(1+uu)})}{uu(uu+1)}$ — $\frac{-du(1 + \sqrt{(1+uu)})}{uu(uu+1)}$

vnde et his conuenienter reliqui calculi ad hoc exemplum pertinentes ſunt corrigendi.

pag. 54. lin. 7. loco $-\text{Ang. cof} \frac{2ab + (aa + bb) \text{cof.} \Phi)}{aa + bb + ab. \text{cof.} \Phi}$

lege $-\text{Ang. cof.} \frac{2ab + (aa + bb) \text{cof.} \Phi}{aa + bb + 2ab \text{cof.} \Phi}$.

pag. 63. lin. 16. loco $\alpha l x$ lege $\alpha l \frac{x}{a}$.

pag. 65

pag. 65. lin. 1. loco tum lege tam.

lin. 13. loco $\frac{n\,ds}{\beta(s-n)}$ lege $\frac{n\,ds}{s(s-n)}$.

pag. 66. lin. 12. loco $u(p-u)\,dp\,\sqrt{(1+nu)}$
lege $n(p-u)\,dp\,\sqrt{(1+uu)}$.

pag. 67. lin. 2. loco $\frac{d\omega \operatorname{cof.} \Phi}{\operatorname{fin.} \Phi \operatorname{cof.} \omega}$ lege $\frac{d\omega \operatorname{cof.} \Phi}{\operatorname{fin.} \psi \operatorname{cof.} \omega}$.

pag. 71. lin. 6. loco $\frac{d\Phi}{1-n \operatorname{fin.} \psi}$ lege $\frac{d\psi}{1-n \operatorname{fin.} \psi}$.

lin. 9. — tres has — hanc.

pag. 81. lin. 12. loco $+y\,du\,vy\,dx$ lege $+y\,du = vy\,dx$.

pag. 82. lin. 8. loco $e^{ndx}(du+uudx)$ le. $\frac{e^{\int ndx}(du+uudx)}{dx}$

pag. 84. lin. 13. loco opertet lege oportet.

pag. 91. lin. 7. — H — K

pag. 96. lin. 2. loco $Ce^{-\frac{1}{2}A} \operatorname{fin.}(mx+\gamma)$
lege $Ce^{-\frac{1}{2}Ax} \operatorname{fin.}(mx+\gamma)$.

pag. 97. lin. 13. loco $\int \frac{\alpha f e^{(g-f)x} - \beta g}{\alpha e^{(g-f)x} - \beta}\,dx$
lege $-\int \frac{\alpha f e^{(g-f)x} - \beta\gamma}{\alpha e^{(g-f)x} - \beta}\,dx$.

pag. 99. lin. 14. loco paraticulariter l. particulariter

pag. 103. lin. 3. loco $P\,dx\,du$ lege $P\,dv\,dx$.

pag. 106. lin. 19. loco $\frac{\beta x}{x}$ lege $\frac{\beta x}{\alpha}$

pag. 108. lin. 2 loco αe^{-fg} lege αe^{-fx}.

pag. 110. lin. 11. loco $\frac{e^{-2f\operatorname{cof.}\zeta}dx}{vv}$ lege $\frac{e^{-2fx\operatorname{cof.}\zeta}dx}{vv}$.

pag.

pag. 111. lin. 16. loco vidimns lege vidimus.

pag. 112. lin. 15. loco ſcribendo ω lege ſcribendo α.

pag. 113. lin. 6. loco ifitegrale lege integrale.

lin. 12. loco αe^{-fg} lege αe^{-fx}.

pag. 126. lin. 3. loco $y = e^{(\mu-\beta)t} \int e^{(\beta-2\mu)t} dt \int \frac{e^{\mu t} X dx}{x^n}$

lege $y = e^{(\mu-\beta)t} \int e^{(\beta-2\mu)t} x^n dt \int \frac{e^{\mu t} X dx}{x^n}$.

pag. 127. lin. 18. loco $QUdx$ lege $QUdx^2$.

pag. 130. lin. 15. loco $(PP - \frac{5 dP}{dx} + 2Q) dM$

lege $(PP - \frac{5 dP}{a dx} + 2Q) dM$.

pag. 134. lin. 5. loco $\frac{1}{4} y dx (\frac{\mu(\mu-1)}{xx} + \frac{2\mu\nu + \quad + \nu + 1C}{x(a+x)} + \frac{\nu(\nu-1)}{(a+x)^2})$

lege $\frac{1}{4} y dx (\frac{\mu(\mu-1)}{xx} + \frac{2\mu\nu + \mu + \nu + 1C}{x(a+x)} + \frac{\nu(\nu-1)}{(a+x)^2})$

pag. 141. lin. 3. loco $-y(d. \frac{dP}{P} - \frac{dP dR}{PR} + \frac{aaRR dx}{PP})^2$

lege $-y(d. \frac{dP}{P} - \frac{dP dR}{PR} + \frac{aaRR dx^2}{PP})$

pag. 142. lin. 18. loco $\frac{(n+1)^2 aayy}{(aa+xx)^2}$ lege $\frac{(n-1)^2 aayy}{(aa+xx)^2}$

pag. 147. lin. 13. loco N lege y

pag. 152. lin. vlt. loco videantur lege videntur.

pag. 163. lin. 19. loco $4AAxxdx$ lege $3AAxxdx$.

pag. 167. lin. 5. loco qdy lege qdq.

pag. 168. lin. 11. loc. terminarum leg. terminorum

lin. 2. loco $3Ny^2dx^2dy$ lege $2Ny^2dx^2dy$

pag. 170. lin. 2. l. $4ydx^2dy(\beta 2\gamma x)$ l. $4ydx^2dy(\beta+2\gamma x)$

pag. 173. lin. 14. loco $\frac{4}{3}z^{-\frac{1}{3}}ddz$ lege $\frac{4}{3}z^{\frac{1}{3}}ddz$

15. loco $\frac{4}{3}z^{-\frac{5}{3}}ddz$ lege $\frac{4}{3}z^{\frac{5}{3}}ddz$

pag. 178. lin. 5. loco $\frac{d\,L\,dy}{2\,L}$ lege $\frac{d\,L\,dy}{1\,L}$.

pag. 179. lin. 4. loco $-y^{3n}\frac{ds}{a\,x}$ lege $-y^{1n}\frac{ds}{d\,x}$.

pag. 184. lin. 6. loco habens lege habeat.

pag. 185. lin. antep. colllgitur lege colligitur.

pag. 190. lin. 2. loco $\frac{a\,\mathfrak{C}}{x^{4}}$ lege $\frac{a\,\mathfrak{C}}{x^{6}}$.

— — — — 11. loco $\frac{7\,a^{3}\,A}{2^{2}.\,3^{3}.\,4^{2}}$ lege $-\frac{7\,a^{8}\,A}{2^{2}.\,3^{3}.\,4^{2}}$.

pag. 193. lin. 3. loco difiniri lege definiri.

pag. 198. lin. 12. loco $\frac{-(5\lambda-1)(5\lambda-1)C}{(12\lambda-6)c} = \frac{-(5\lambda-2)(7\lambda-1)C}{6(2\lambda-1)c}$

lege $\frac{-(5\lambda-2)(5\lambda-1)C}{(12\lambda-6)c} = \frac{-(5\lambda-2)(5\lambda-1)C}{6(2\lambda-1)c}$

pag. 204. lin. vlt loco $+\frac{i(ii-2)(i-1)}{2.4ccit}$ lege $+\frac{i(ii-1)(i-2)}{2.4ccit}$

pag. 206. lin. 15. loco $-C\alpha\,\text{fin}.bt$ lege $-\alpha\,\text{fin}.bt$.

pag. 207. lin. 11. loco aequationem differentialem

leg. integrale aequationis differentialis.

pag. 208. lin. 4. loco $-(\alpha e^{ct}+\beta e^{-ct})$ lege $-(\alpha e^{ct}-\beta e^{-ct})$.

— — — — 6. loco $y(\alpha e^{ct}+\beta e^{-ct})P$ l. $y=(\alpha e^{ct}+\beta e^{-ct})P$

— — — — vlt. loco $d\,Q$ lege $\frac{dQ}{dt}$.

pag. 209. lin. 3. loco $\frac{i(ii-1)(i-2)}{2.4}t^{i-2}$ lege $\frac{i(ii-1)(i-2)}{2.4\,c}t^{i-2}$.

— — — — vlt. loco abrumpantur l. abrumpuntur.

pag. 211. lin. 5. loco $P=\frac{1}{6\,t^{6}}+\text{etc.}$ lege $P=\frac{1}{t^{6}}+\text{etc.}$

pag. 211.

pag. 211. lin. penult. loco $\frac{3.5.5}{cc\,l^8}$ lege $\frac{3.5.5}{cc\,l^6}$.

— — — — vlt. loco $\frac{3.5.7}{c^7\,l^7}$ lege $\frac{3.5.7}{c^8\,l^7}$.

pag 212. lin. 3. loco $\frac{10}{c\,l^8}$ lege $\frac{10}{c\,l^6}$.

pag. 217. lin. 11. loco $\frac{1}{ll}$ lege $\frac{1}{l^3}$.

pag. 218. lin. 5. loco $-\frac{3.5.7\,l^2}{b^2}$ lege $-\frac{3.5.7\,l^2}{b^8}$.

pag. 230. lin. 7. loco $2n((2n+2\lambda-1)a+e)\mathfrak{C}$

lege $2n((2n+2\lambda-1)a+c)\mathfrak{C}$.

pag. 231. lin. 5. loco $-4bf$ lege $-4af$.

pag. 233. lin. 19. loco logarithmus l. logarithmum

pag. 235. lin. 14. loco $\frac{548\,A\,g^5}{1.7.27.64.125.n}ii$ lege $\frac{548\,A\,g^5}{1.8.27.64.125.n}ii$.

— — — — penult. loco $+\frac{\alpha g^4}{2.4.9.10n}x^{4n}$ l. $+\frac{\alpha g^4}{2.4.9.10n}x^{4n}$

pag. 237. lin. 18. loco $\frac{5.7}{4.6}$C lege $\frac{5.7}{6.8}$C.

— — — lin. 21. loco $\frac{3.5}{4.6}\mathfrak{C}$ lege $\frac{3.5}{4.6}\mathfrak{C}$.

pag. 238. lin. 15. loco $n=\mathfrak{A}x+$etc. lege $u=\mathfrak{A}x+$etc.

pag. 239. lin. 4. loco $6\,\mathfrak{C}$ lege $5\,\mathfrak{C}$.

— — — lin. 20. loco $\mathfrak{B}(2.3b+2e+g)$

lege $\mathfrak{B}(2.3b+3e+g)$.

pag. 240. lin. 12. loco $bb\mathfrak{B}=0$ lege ob $\mathfrak{B}=0$.

pag. 241. lin. 14. loco $A=\frac{1}{2}\mathfrak{A}$etc. lege $A=-\frac{1}{2}\mathfrak{A}$etc.

pag. 243. lin. 2. loco $\frac{-C}{2x^2}$ lege $\frac{-bC}{2x^2}$.

pag. 249. lin. 12. loco est lege et.

pag. 250. lin. 12. loco $-(2(i-1)n-1)$ l. $-(2(i-1)n-1)b$

Xxx 2 p. 253.

pag. 253. lin. 7. loco ideoque lege idque.

— — — — 9. loco ddy lege $Lddy$.

pag. 257. lin. 4. deleatur *seu.*

pag. 266. lin. 20. loco $2Dx^2$ lege $2Dax^2$.

pag. 280. lin. 2. loco *cnm* lege *cum.*

pag. 281. lin. 9. loco adhunc lege adhuc.

pag. 283. lin. 12. loco $\frac{y}{du}$ lege $\frac{dy}{du}$.

pag. 284. lin. 14. loco elilicitur lege elicitur.

pag. 286. lin. 19. loco etiam lege etiamsi.

pag. 287. lin. 6. loco $(uu+xx)^{\mu-1}$ lege $(uu+xx)^{\mu-2}$.

pag. 288. lin. 7. loco $(n+4\mu+2\nu+1)uu$

lege $(n+4\mu+2\nu-1)uu$.

— — — — 11. loco $\frac{(n-2\mu-1)(cc+uu)dy}{2\mu u du}$

lege $\frac{(n+2\mu-1)(cc+uu)dy}{2\mu u du}$.

pag. 289. lin. 7. loco $(cc+xx)^{\frac{\beta-1+2n-2\alpha}{2}}$

lege $(cc-xx)^{\frac{\beta-1+n-2\alpha}{2}}$

lin. 8. loco $\beta+1+n>2\alpha$ l. $\beta-1+n>2\alpha$.

lin. 15. loco §. 998. lege §. 997.

pag. 291. lin. vlt. loco $\frac{(n-\nu)r^{\frac{-2n+2\nu-1}{n+\nu+1}}dr}{2(n+\nu+1)}$

lege $\frac{(n-\nu)r^{\frac{-2n-2\nu-1}{n+\nu+1}}dr}{2(n+\nu+1)}$

pag. 292.

pag. 292. lin. 2. loco $aar^{\frac{-2n-2\nu-4}{n+\nu+1}} dr$ lege $aar^{\frac{-3n-3\nu-4}{n+\nu+1}} dr$

pag. 294. lin. 18. loco $\frac{ddy}{dx^2}$ lege $\frac{ddy}{du^2}$.

pag. 307. lin. 7. loco $\frac{(A+\alpha v)ddy}{dv^2}$ lege $\frac{(A+\alpha u)ddy}{du^2}$.

pag. 315. lin. 2. loco $D = \frac{2m+b}{2n+k}C$ lege $D = \frac{2m+b}{3n+k}C$

pag. 316. lin. 9. loco $\frac{(i-1)\mu+\eta}{i\nu+\theta} \int P\, dx^{i-1}\, dx$

lege $\frac{(i-1)\mu+\eta}{i\nu+\theta} \int P\, x^{i-1}\, dx$.

pag. 318. lin. 3. loco $y/x^{\frac{\eta}{\mu}-1} dx(\mu-\nu x)^{\frac{\theta\mu+\eta\nu}{\mu\nu}}$

lege $y\int x^{\frac{\eta}{\mu}-1} dx(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$.

lin. 17. loco hoc lege hac

pag. 321. lin. 3. loco $-(i-1)^2 nub$ lege $-(i-1)^2 mb$

pag. 322. lin. 11. loco §. 1064. lege §. 1063.

lin. 16. loco $(\mu-\nu t)^{\frac{\theta\mu+\eta\nu}{\mu\nu}}$ lege $(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}$.

pag. 323. lin. 5. loco $\frac{(2\lambda-1)b+c+q}{2nb}$ lege $\frac{(2\lambda-1)b+c-q}{2nb}$.

pag. 324. lin. vlt. in coefficiente ipsius M, elidatur $+(i-1)na$.

pag. 325. lin. 11. loco $b = -\frac{1}{2}(2\lambda-1)b - \frac{1}{2}c - \frac{1}{2}p$

lege $b = -\frac{1}{2}(2\lambda-1)a - \frac{1}{2}c - \frac{1}{2}p$.

pag. 328. lin 16. deleatur *ita.*

pag. 335. lin. 10. loco ostendum lege ostendendum

pag. 340. lin. 5. loco $\frac{D}{x^{\nu}(n-x)^{\mu}}$ lege $\frac{D}{x^{\nu}(a-x)^{\mu}}$.

pag. 343. lin. 1. loco veniri lege venire.

pag. 344. lin penult. loco $y=b+\frac{c\Phi}{}$ lege $y=b+\frac{c\Phi}{P}$.

pag. 347. lin. 12. loco infinitia lege infinita.

pag. 348. lin. 3. loco $p=\lambda A(-a)^{\lambda-1}$ lege $p=\lambda A(x-a)^{\lambda-1}$

pag. 351. lin, 13. loco exiguae lege exigua.

pag. 352. lin. 15. loco difiniuimus lege definiuimus

pag. 356. lin. 12. loco ddx lege ddy.

pag. 382. lin. 8. loco prodit lege dat.

pag. 420. lin. 9. loco $\frac{}{\alpha\cdot-\delta}$ lege $\frac{1}{\alpha-\delta}$.

pag. 423. lin. 6. loco $3\,D\nu^{2}$ lege $3\,D\nu^{2}$.

pag. 440. lin. 11. loco $-4Ef^{3}\text{ſin.}\,2\,\theta$ l. $-4Ef^{3}\text{ſin.}\,3\,\theta$

pag. 446. lin. 10. loco $\pm\frac{n'-1)(n-2)}{1.\,2.\,3}Nf^{n-3}\text{coſ.}(n-3)\theta$

lege $\pm\frac{n(n-1)(n-2)}{1.\,2.\,3}Nf^{n-3}\text{coſ}(n-3)\theta$.

lin. 11. loco $\pm\frac{n(-1)(n-2)}{1.\,2.\,3}$ etc.

lege $\pm\frac{n(n-1)(n-2)}{1.\,2.\,3}$ etc.

pag. 449. lin. vlt. loco expoſita lege expoſito.

pag. 470. lin. 11. loco $+\frac{ddx}{6\,dx^{2}}$ lege $\frac{ddv}{6\,a\,x^{2}}$.

pag. 476. lin. 18. loco $\frac{d^{8}}{1\ldots6\,c^{6}}$ lege $\frac{z^{8}}{1\ldots6\,c^{6}}$.

pag. 483. lin. vlt. loco $Xx\lambda dx$ lege $Xx^{\lambda}dx$.